

DYNAMICS FOR THE FOCUSING, ENERGY-CRITICAL NONLINEAR HARTREE EQUATION

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ABSTRACT. In [41, 51], the dynamics of the solutions for the focusing energy-critical Hartree equation have been classified when $E(u_0) < E(W)$, where W is the ground state. In this paper, we continue the study on the dynamics of the radial solutions with the threshold energy. Our arguments closely follow those in [16, 17, 18, 43, 44]. The new ingredient is that we show that the positive solution of the nonlocal elliptic equation in $L^{\frac{2d}{d-2}}(\mathbb{R}^d)$ is regular and unique by the moving plane method in its global form, which plays an important role in the spectral theory of the linearized operator and the dynamics behavior of the threshold solution.

1. INTRODUCTION

We consider the focusing, energy-critical Hartree equation

$$i\partial_t u + \Delta u + (|x|^{-4} * |u|^2) u = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^d \quad (1.1)$$

where $d \geq 5$ in this context. The Hartree equation arises in the study of Boson stars and other physical phenomena, please refer to [61]. In chemistry, it appears as a continuous-limit model for the mesoscopic structures, see [22]. Moreover, (1.1) enjoys several symmetries: If $u(t, x)$ is a solution, then

- (a) by scaling: so is $\lambda^{-\frac{d-2}{2}} u(\lambda^{-2}t, \lambda^{-1}x)$, $\lambda > 0$;
- (b) by time translation invariance: so is $u(t + t_0, x)$ for $t_0 \in \mathbb{R}$;
- (c) by spatial translation invariance: so is $u(t, x + x_0)$ for $x_0 \in \mathbb{R}^d$;
- (d) by phase rotation invariance: so is $e^{i\theta_0} u(t, x)$, $\theta_0 \in \mathbb{R}$;
- (e) by time reversal invariance: so is $\overline{u(-t, x)}$.

The local well-posedness for the Cauchy problem of (1.1) was developed in [7, 50]. Namely, if $u_0 \in \dot{H}^1(\mathbb{R}^d)$, there exists a unique solution defined on a maximal interval $I = (-T_-(u), T_+(u))$ and the energy

$$E(u(t)) := \frac{1}{2} \int_{\mathbb{R}^d} |\nabla u(t, x)|^2 dx - \frac{1}{4} \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|u(t, x)|^2 |u(t, y)|^2}{|x - y|^4} dx dy = E(u_0) \quad (1.2)$$

is conserved on I . The name “energy critical” refers to the fact that the scaling

$$u(t, x) \rightarrow u_\lambda(t, x) = \lambda^{-\frac{d-2}{2}} u(\lambda^{-2}t, \lambda^{-1}x), \quad \lambda > 0 \quad (1.3)$$

makes the equation (1.1) and its energy (1.2) invariant.

There are many results for the energy-critical Hartree equation. For the defocusing case, Miao, etc, take advantage of the term $-\int_I \int_{|x| \leq A|I|^{1/2}} |u|^2 \Delta \left(\frac{1}{|x|} \right) dx dt$ in the localized Morawetz

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identity, which is related to the linear operator $i\partial_t + \Delta$ to rule out the possibility of energy concentration, instead of the classical Morawetz estimate dependent of the nonlinearity and thus obtain the global well-posedness and scattering of the radial solution in [49]. Subsequently, Miao, etc use the induction on energy argument in both the frequency space and the spatial space simultaneously and the frequency-localized interaction Morawetz estimate to remove the radial assumption in [53].

For the focusing case, the dynamics behavior becomes complicated. It turns out that the explicit ground state

$$W(x) = c_0 \left(\frac{t}{t^2 + |x|^2} \right)^{-\frac{d-2}{2}} \quad \text{with } c_0 > 0, t > 0 \quad (1.4)$$

plays an important role in the dynamical behavior of solutions for (1.1). Miao, etc, make use of the concentration compactness principle and the rigidity argument, which are first introduced in NLS and NLW by C. Kenig and F. Merle in [30, 31], to show that

Theorem 1.1 ([41, 51]). *Let u be a solution of (1.1) with*

$$u_0 \in \dot{H}^1(\mathbb{R}^d), \quad E(u) < E(W).$$

Then

- (a) *if $\|\nabla u_0\|_{L^2} < \|\nabla W\|_{L^2}$, then $I = \mathbb{R}$ and $\|u\|_{L_t^6(L_x^{\frac{6d}{3d-8}})} < \infty$;*
- (b) *if $\|\nabla u_0\|_{L^2} > \|\nabla W\|_{L^2}$, and either $u_0 \in L^2$ is radial or $x \cdot u_0 \in L^2$, then $T_{\pm} < \infty$.*

For other dynamics results of the Hartree equation, please refer to [6, 25, 35, 36, 41, 42, 50, 51, 52, 54, 55, 56].

In this paper, we continue the study on the dynamics of the radial solutions with the threshold energy $E(W)$. Our goal is to give the classification of the solutions, that is, the initial data $u_0 \in \dot{H}^1(\mathbb{R}^d)$ is radial and satisfies

$$E(u_0) = E(W). \quad (1.5)$$

In this case, the classification is more abundant than that in Theorem 1.1. Clearly, W is a new solution which doesn't satisfy neither conclusion in Theorem 1.1. Besides W , there also exist two other special radial solutions W^{\pm} .

Theorem 1.2. *There exist two radial solutions W^{\pm} of (1.1) with initial data W_0^{\pm} such that*

- (a) $E(W^{\pm}) = E(W)$, $T_+(W^{\pm}) = \infty$ and $\lim_{t \rightarrow +\infty} W^{\pm}(t) = W$ in \dot{H}^1 .
- (b) $\|\nabla W_0^-\|_2 < \|\nabla W\|_2$, $T_-(W^-) = \infty$ and W^- scatters for the negative time.
- (c) $\|\nabla W_0^+\|_2 > \|\nabla W\|_2$, and $T_-(W^+) < \infty$.

Next, we characterize all radial solutions with the threshold energy as follows:

Theorem 1.3. *Suppose $u_0 \in \dot{H}^1(\mathbb{R}^d)$ is radial, and such that*

$$E(u_0) = E(W).$$

Let u be a solution of (1.1) with initial data u_0 and I be the maximal interval of existence. Then the following holds:

- (a) *If $\|\nabla u_0\|_2 < \|\nabla W\|_2$, then $I = \mathbb{R}$. Furthermore, either $u = W^-$ up to the symmetries of the equation, or $\|u\|_{L_t^6 L_x^{\frac{6d}{3d-8}}} < \infty$.*

- (b) If $\|\nabla u_0\|_2 = \|\nabla W\|_2$, then $u = W$ up to the symmetries of the equation.
- (c) If $\|\nabla u_0\|_2 > \|\nabla W\|_2$, and $u_0 \in L^2(\mathbb{R}^d)$, then either $u = W^+$ up to the symmetries of the equation, or I is finite.

Because of the radial assumption, the symmetries of the equation in the above theorem refer to the symmetries under scaling, time translation, phase rotation and time reverse.

Now we begin with a brief recapitulation of some important dynamics results for NLS, NLW and NLKG that have been derived so far. The orbital stability of the soliton for the L^2 -subcritical NLS in the energy space was settled by Weinstein [69, 70], Berestycki and Cazenave [4], Cazenave and Lions [9]. In detail, Weinstein obtained the quantitative analysis (modulation stability analysis), while Berestycki, Cazenave and Lions gave the qualitative analysis. After the successful applications of the concentration compactness principle (the profile decomposition [3, 28, 32]) into the global existence and scattering theory for the \dot{H}^1 -critical NLS and NLW with the energy less than that of the ground state in [30, 31], Duyckaerts and Merle combined the spectral theory of the linearized operator, the modulational stability of the soliton with the concentration compactness argument to classify the solutions with the threshold energy for the \dot{H}^1 -critical NLS and NLW in [16, 17]. Subsequently, Duyckaerts and Roudenko dealt with the 3D cubic NLS case in [18]. Li and Zhang obtained the dynamics of threshold solutions for the focusing, \dot{H}^1 -critical NLS and NLW in the higher dimensions in [43, 44]. For the more recent progresses on the global dynamics above the ground state energy of NLS, NLW and NLKG, please refer to [37, 38, 57, 58, 59, 60].

The paper is organized as follows. The main structure of the paper is reminiscent of that for the NLS and NLW in [16, 17, 18, 43, 44]. The new ingredient is that we show that the positive solution of the nonlocal elliptic equation in $L^{\frac{2d}{d-2}}(\mathbb{R}^d)$ is regular and unique by the moving plane method in its global form, which plays an important role in the spectral theory of the linearized operator and the dynamics behavior of the threshold solution.

In Section 2, we recall the Cauchy problem, the properties of the ground state. We also state the spectral properties of the linearized operator \mathcal{L} around W , which is deduced from the property of the null space of the linearized operator in L^2_{rad} ; Under the condition $E(u_0) = E(W)$, we can identify a quadratic form B associated to the linearized operator \mathcal{L} and use the property of the null space of the linearized operator to find two subspaces $H^\perp \cap \dot{H}^1_{rad}$ and $G_\perp \cap \dot{H}^1_{rad}$ in \dot{H}^1 , where the linearized energy Φ is positive (coercive), avoiding the vanishing and negative directions. These decompositions will play an important role in establishing the modulational stability in Section 4 and analyzing the uniqueness of the exponential decaying solutions to the linearized equation in Section 7, respectively.

In Section 3, we construct two special solutions W^\pm of (1.1) except for the negative time behavior by use of the knowledge about the real eigenvalues of the linearized operator \mathcal{L} and the fixed point argument, which gives the proof of Theorem 1.2 except for the negative time behavior.

In Section 4, we make use of the variational characterization of W and the implicit function theorem to discuss the modulational stability around W , then we make use of the positivity of the linearized energy Φ in $H^\perp \cap \dot{H}^1_{rad}$ to identify the scaling and phase parameters in the modulational stability which are closely related with the gradient variant $\delta(t)$ of the solution away from W . In particular, these parameters are linearly dependent of $\delta(t)$ in the interval with small gradient variant.

In Section 5 and Section 6, we study the solutions with initial data satisfying Theorem 1.3 part (a) and (c). Main techniques are to make use of the virial argument and the concentration compactness argument to obtain the exponential decay (5.13) and (6.20) of the gradient

variant $\delta(t)$ for the large (positive) time, which will imply the exponential convergence in the (positive) time direction to W (up to scaling and phase rotation) by the modulational stability, and also obtain the proof of Theorem 1.2 for the negative time behavior.

In Section 7, we first use the positivity of the linearized energy Φ in $G_\perp \cap \dot{H}_{rad}^1$ to analyze the property of the exponentially decay solution of the linearized equation, then apply it to establishing the uniqueness of the special solutions, this will imply the proof of Theorem 1.3.

Appendix A contains proofs of the uniqueness of the ground state in $L^{\frac{2d}{d-2}}$ by the moving plane method in its global form. Appendix B and C contain proofs of the spectral properties and positivity of the linearized operator in Proposition 2.14 and Proposition 2.15.

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2. PRELIMINARIES

2.1. The linear estimates and the Cauchy problem. In this section, we recall some results on the Cauchy problem of (1.1). Let I be an interval, and denote

$$Z(I) := L^6 \left(I; L^{\frac{6d}{3d-8}}(\mathbb{R}^d) \right), \quad S(I) := L^3 \left(I; L^{\frac{6d}{3d-4}}(\mathbb{R}^d) \right), \quad N(I) := L^{\frac{3}{2}} \left(I; L^{\frac{6d}{3d+4}}(\mathbb{R}^d) \right),$$

$$l(I) = Z(I) \cap L^3 \left(I; \dot{W}^{1, \frac{6d}{3d-4}}(\mathbb{R}^d) \right), \quad \|u\|_{l(I)} := \|u\|_{Z(I)} + \|\nabla u\|_{S(I)}.$$

A solution of (1.1) on an interval I with $0 \in I$ is a function $u \in C^0(I, \dot{H}^1(\mathbb{R}^d))$ such that $u \in Z(J)$ for all interval $J \Subset I$ and

$$u(t) = e^{it\Delta} u_0 + i \int_0^t e^{i(t-s)\Delta} \left(\frac{1}{|\cdot|^4} * |u(s, \cdot)|^2 \right) (x) u(s, x) ds.$$

Lemma 2.1 ([7, 67]). *Consider*

$$\begin{cases} i\partial_t u + \Delta u = f, & x \in \mathbb{R}^d, t \in [0, T), \\ u(0) = u_0 \in \dot{H}^1, \end{cases} \quad (2.1)$$

where $\nabla f \in N(0, T)$, then we have

$$\sup_{t \in [0, T)} \|u\|_{\dot{H}^1} + \|u\|_{l(0, T)} \leq C \left(\|u_0\|_{\dot{H}^1} + \|\nabla f\|_{N(0, T)} \right).$$

Lemma 2.2 ([26, 63]). *For $\alpha \in (0, d)$, there exists a constant $C(d, \alpha)$ such that for any $r \in (\frac{d}{d-\alpha}, \infty)$,*

$$\left\| \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{d-\alpha}} dy \right\|_{L^r(\mathbb{R}^d)} \leq C(d, \alpha) \|f\|_{L^{\frac{dr}{d+\alpha r}}(\mathbb{R}^d)}.$$

Lemma 2.3 ([3, 23, 34]). *For every $f \in \dot{H}^1(\mathbb{R}^d)$, there exists a constant C such that*

$$\|f\|_{L^{\frac{2d}{d-2}}(\mathbb{R}^d)} \leq C \|f\|_{\dot{H}^1(\mathbb{R}^d)}^{\frac{d-2}{d}} \|f\|_{\dot{B}_{2,\infty}^1(\mathbb{R}^d)}^{\frac{2}{d}}.$$

Theorem 2.4 ([7, 50]). *For any $u_0 \in \dot{H}^1(\mathbb{R}^d)$ and $t_0 \in \mathbb{R}$, there exists a unique maximal-lifespan solution $u : I \times \mathbb{R}^d \rightarrow \mathbb{C}$ to (1.1) with $u(t_0) = u_0$. This solution also has the following properties:*

- (a) I is an open neighborhood of t_0 .

(b) The energy of the solution u are conserved, that is, for all $t \in I$, we have

$$E(u(t)) = E(u_0).$$

- (c) If $u_0^{(n)}$ is a sequence converging to u_0 in \dot{H}^1 and $u^{(n)} : I^{(n)} \times \mathbb{R}^d \rightarrow \mathbb{C}$ are the associated maximal-lifespan solutions, then $u^{(n)}$ converges locally uniformly to u .
- (d) There exists η_0 , such that if $\|u_0\|_{\dot{H}^1} < \eta_0$, then u is a global solution. Indeed, the solution also scatters to 0 in \dot{H}^1 .

2.2. Properties of the ground state. Consider the nonlocal elliptic equation

$$-\Delta W(x) = \int_{\mathbb{R}^d} \frac{|W(y)|^2}{|x-y|^4} dy W(x), \quad x \in \mathbb{R}^d. \quad (2.2)$$

(1.4) is an explicit solution of (2.2). Using the moving plane method in its global form, we will show that W is the only positive solution of (2.2) in $L^{\frac{2d}{d-2}}(\mathbb{R}^d)$ in Appendix A, up to the symmetries of (2.2). Moreover, the uniqueness still holds for the positive solution in $L_{loc}^{\frac{2d}{d-2}}(\mathbb{R}^d)$. Hence, we have

Lemma 2.5. *The elliptic equation (2.2) has a unique positive, radial decreasing solution $W(x) = W(|x|)$ in $L^{\frac{2d}{d-2}}(\mathbb{R}^d)$, up to the spatial translation, scaling and the Kelvin transform.*

Combining the sharp Sobolev inequality [2, 47, 66] with the sharp Hardy-Littlewood-Sobolev inequality [46, 47], we have the following variational characterization of W .

Lemma 2.6 ([47]). *For any $\omega \in \dot{H}^1(\mathbb{R}^d)$, we have*

$$\left(\iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{1}{|x-y|^4} |\omega(x)|^2 |\omega(y)|^2 dx dy \right)^{1/4} \leq C_* \|\nabla \omega\|_2,$$

where $C_* = C_*(d)$ is the best constant. Moreover if

$$\left(\iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{1}{|x-y|^4} |\omega(x)|^2 |\omega(y)|^2 dx dy \right)^{1/4} = C_* \|\nabla u\|_2,$$

then there exist $\lambda_0 > 0, x_0 \in \mathbb{R}^d, \theta_0 \in [0, 2\pi)$, such that

$$\omega(x) = e^{i\theta_0} \lambda_0^{-\frac{d-2}{2}} W(\lambda_0^{-1}(x+x_0)).$$

From (2.2) and Lemma 2.6, we have

$$\|\nabla W\|_2^2 = C_*^{-4}, \quad E(W) = C_*^{-4}/4. \quad (2.3)$$

Using the characterization of W in Lemma 2.6, the refined Sobolev inequality in Lemma 2.3 and the similar concentration compactness principle (profile decomposition in \dot{H}_{rad}^1) to the proof of Proposition 3.1 in [55], we can show that

Proposition 2.7. *Let $u \in \dot{H}^1(\mathbb{R}^d)$ be radial and $E(u) = E(W)$. Then there exists a function $\varepsilon = \varepsilon(\rho)$, such that*

$$\inf_{\theta \in \mathbb{R}, \mu > 0} \|u_{\theta, \mu} - W\|_{\dot{H}^1} \leq \varepsilon(\delta(u)), \quad \lim_{\rho \rightarrow 0} \varepsilon(\rho) = 0,$$

where $u_{\theta, \mu}(x) = e^{i\theta} \mu^{-\frac{d-2}{2}} u(\mu^{-1}x)$, and $\delta(u) = \left| \int_{\mathbb{R}^d} (|\nabla u|^2 - |\nabla W|^2) dx \right|$.

Proof. Suppose the contrary holds. Let $u_n \in \dot{H}^1$ be any radial sequence and satisfy

$$E(u_n) = E(W), \text{ and } \left| \int_{\mathbb{R}^d} (|\nabla u_n|^2 - |\nabla W|^2) dx \right| \longrightarrow 0. \quad (2.4)$$

Hence, there exists a subsequence of $\{u_n\}_{n=1}^\infty$ and a sequence $\{U^{(j)}\}_{j \geq 1}$ in \dot{H}_{rad}^1 and for any $j \geq 1$, a family $\{\lambda_n^j\}$ such that

(1) If $j \neq k$, we have the asymptotic orthogonality between $\{\lambda_n^j\}$ and $\{\lambda_n^k\}$, i.e.

$$\frac{\lambda_n^j}{\lambda_n^k} + \frac{\lambda_n^k}{\lambda_n^j} \longrightarrow +\infty, \quad \text{as } n \rightarrow +\infty.$$

(2) For every $l \geq 1$, we have

$$u_n(x) = \sum_{j=1}^l \frac{1}{(\lambda_n^j)^{(d-2)/2}} U^{(j)}\left(\frac{x}{\lambda_n^j}\right) + r_n^l(x), \text{ with } \lim_{l \rightarrow +\infty} \limsup_{n \rightarrow +\infty} \|r_n^l\|_{\dot{B}_{2,\infty}^1} \rightarrow 0.$$

(3) We have

$$\|\nabla u_n\|_{L^2}^2 = \sum_{j=1}^l \|\nabla U_n^{(j)}\|_{L^2}^2 + \|\nabla r_n^l\|_{L^2}^2 + o_n(1).$$

By Lemma 2.2, Lemma 2.3 and the orthogonality between λ_n^j and λ_n^k for $j \neq k$, we have

$$\begin{aligned} \lim_{l \rightarrow +\infty} \limsup_{n \rightarrow +\infty} \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|r_n^l(x)|^2 |r_n^l(y)|^2}{|x-y|^4} dx dy &= 0, \\ \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|u_n(x)|^2 |u_n(y)|^2}{|x-y|^4} dx dy &= \lim_{j \rightarrow +\infty} \sum_{j=1}^l \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|U^{(j)}(x)|^2 |U^{(j)}(y)|^2}{|x-y|^4} dx dy. \end{aligned}$$

By Lemma 2.6, we have

$$C_*^{-4} \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|U^{(j)}(x)|^2 |U^{(j)}(y)|^2}{|x-y|^4} dx dy \leq \|\nabla U^{(j)}\|_{L^2}^4,$$

thus,

$$C_*^{-4} \sum_{j=1}^l \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|U^{(j)}(x)|^2 |U^{(j)}(y)|^2}{|x-y|^4} dx dy \leq \sum_{j=1}^l \|\nabla U^{(j)}\|_{L^2}^4.$$

This yields that

$$C_*^{-4} \leq \lim_{n \rightarrow +\infty} \limsup_{l \rightarrow +\infty} \frac{\sum_{j=1}^l \|\nabla U^{(j)}\|_{L^2}^4}{\sum_{j=1}^l \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|U^{(j)}(x)|^2 |U^{(j)}(y)|^2}{|x-y|^4} dx dy}$$

$$\begin{aligned}
& \leq \lim_{n \rightarrow +\infty} \limsup_{l \rightarrow +\infty} \frac{\left(\sum_{j=1}^l \|\nabla U^{(j)}\|_{L^2}^2 \right)^2}{\sum_{j=1}^l \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|U^{(j)}(x)|^2 |U^{(j)}(y)|^2}{|x-y|^4} dx dy} \\
& \leq \lim_{n \rightarrow +\infty} \frac{\|\nabla u_n\|_{L^2}^4}{\iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|u_n(x)|^2 |u_n(y)|^2}{|x-y|^4} dx dy} = C_*^{-4},
\end{aligned}$$

where we use (2.4) in the last step. Therefore, we conclude that only one profile $U^{(j)}$ is nonzero, and

$$C_*^{-4} = \frac{\|\nabla U^{(j_0)}\|_{L^2}^4}{\iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|U^{(j_0)}(x)|^2 |U^{(j_0)}(y)|^2}{|x-y|^4} dx dy}.$$

By Lemma 2.6, there exist $\theta_0 \in [0, 2\pi)$, $\lambda_0 > 0$ such that

$$U^{(j_0)}(x) = e^{i\theta_0} \lambda_0^{-\frac{d-2}{2}} W(\lambda_0^{-1}x).$$

This contradicts the assumption and we completes the proof. \square

2.3. The gradient separation. By the convex analysis in [51], we first have

$$\begin{aligned}
\forall u \in \dot{H}^1, \quad E(u) \leq E(W), \quad \|\nabla u\|_2 \leq \|\nabla W\|_2 &\implies \frac{\|\nabla u\|_2^2}{\|\nabla W\|_2^2} \leq \frac{E(u)}{E(W)}, \\
\forall u \in \dot{H}^1, \quad E(u) \leq E(W), \quad \|\nabla u\|_2 \geq \|\nabla W\|_2 &\implies \frac{\|\nabla u\|_2^2}{\|\nabla W\|_2^2} \geq \frac{E(u)}{E(W)}.
\end{aligned} \tag{2.5}$$

This, together with the energy conservation, the variational characterization of W and the continuity argument, implies that

Lemma 2.8. *Let $u \in \dot{H}^1(\mathbb{R}^d)$ be a radial solution of (1.1) with initial data u_0 , and $I = (-T_-, T_+)$ its maximal interval of existence. Assume that $E(u_0) = E(W)$, then*

- (a) *if $\|\nabla u_0\|_2 < \|\nabla W\|_2$, then $\|\nabla u(t)\|_2 < \|\nabla W\|_2$ for $t \in I$.*
- (b) *if $\|\nabla u_0\|_2 = \|\nabla W\|_2$, then $u = W$ up to the symmetry of the equation.*
- (c) *if $\|\nabla u_0\|_2 > \|\nabla W\|_2$, then $\|\nabla u(t)\|_2 > \|\nabla W\|_2$ for $t \in I$.*

Proof. The proof is analogue to that of Proposition 3.1 in [51]. \square

2.4. Monotonicity formula. Let $\phi(x)$ be a smooth radial function such that $\phi(x) = |x|^2$ for $|x| \leq 1$ and $\phi(x) = 0$ for $|x| \geq 2$. For $R > 0$, define

$$V_R(t) = \int_{\mathbb{R}^d} \phi_R(x) |u(t, x)|^2 dx, \quad \text{where } \phi_R(x) = R^2 \phi\left(\frac{x}{R}\right). \tag{2.6}$$

Lemma 2.9 ([41, 51]). *Let $u(t, x)$ be a radial solution to (1.1), $V_R(t)$ be defined by (2.6), then*

$$\partial_t V_R(t) = 2\Im \int_{\mathbb{R}^d} \bar{u} \nabla u \cdot \nabla \phi_R dx,$$

$$\partial_t^2 V_R(t) = 8 \int_{\mathbb{R}^d} |\nabla u(t, x)|^2 dx - 8 \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|u(t, x)|^2 |u(t, y)|^2}{|x - y|^4} dx dy + A_R(u(t)),$$

where

$$\begin{aligned} A_R(u(t)) &= \int_{\mathbb{R}^d} \left(4\phi''\left(\frac{|x|}{R}\right) - 8 \right) |\nabla u(t, x)|^2 dx + \int_{\mathbb{R}^d} (-\Delta \Delta \phi_R(x)) |u(t, x)|^2 dx \\ &\quad + 8 \iint_{\mathbb{R}^d \times \mathbb{R}^d} \left(1 - \frac{1}{2} \frac{R}{|x|} \phi'\left(\frac{|x|}{R}\right) \right) \frac{x(x-y)}{|x-y|^6} |u(t, x)|^2 |u(t, y)|^2 dx dy \\ &\quad - 8 \iint_{\mathbb{R}^d \times \mathbb{R}^d} \left(1 - \frac{1}{2} \frac{R}{|y|} \phi'\left(\frac{|y|}{R}\right) \right) \frac{y(x-y)}{|x-y|^6} |u(t, x)|^2 |u(t, y)|^2 dx dy. \end{aligned}$$

2.5. Preliminary properties of the linearized operator. We consider a radial solution u of (1.1) close to W and write u as

$$u(t, x) = W(x) + h(t, x),$$

then h satisfies that

$$i\partial_t h + \Delta h = Vh + iR(h),$$

where the linear operator V and the remainder $R(h)$ are defined by

$$Vh := -\left(\frac{1}{|x|^4} * |W|^2\right)h - 2\left(\frac{1}{|x|^4} * (W\Re h)\right)W, \quad (2.7)$$

$$R(h) := i\left(|\cdot|^{-4} * |h|^2\right)(W + h) + 2i\left(|\cdot|^{-4} * (W\Re h)\right)h. \quad (2.8)$$

In the following context, we will always denote the complex value function $h = h_1 + ih_2 = (h_1, h_2)$ without confusion. Let $h_1 = \Re h$ and $h_2 = \Im h$. Then h is a solution of the equation

$$\partial_t h + \mathcal{L}h = R(h), \quad \mathcal{L} := \begin{pmatrix} 0 & -L_- \\ L_+ & 0 \end{pmatrix}, \quad (2.9)$$

where the self-adjoint operators L_{\pm} are defined by

$$L_+ h_1 := -\Delta h_1 - \left(\frac{1}{|x|^4} * |W|^2\right)h_1 - 2\left(\frac{1}{|x|^4} * (W h_1)\right)W, \quad (2.10)$$

$$L_- h_2 := -\Delta h_2 - \left(\frac{1}{|x|^4} * |W|^2\right)h_2. \quad (2.11)$$

Now we first give some preliminary estimates about the linearized equation (2.9).

Lemma 2.10. *Let V , R be defined by (2.7) and (2.8), respectively, I be a interval with $|I| \leq 1$, and $g, h \in l(I)$, $u, v \in L^{\frac{2d}{d-2}}(\mathbb{R}^d)$, then*

$$\|\nabla(Vh)\|_{N(I)} \lesssim |I|^{\frac{1}{3}} \|h\|_{l(I)}, \quad (2.12)$$

$$\|\nabla(R(g) - R(h))\|_{N(I)} \lesssim \|g - h\|_{l(I)} \left(|I|^{\frac{1}{3}} (\|g\|_{l(I)} + \|h\|_{l(I)}) + \|g\|_{l(I)}^2 + \|h\|_{l(I)}^2 \right), \quad (2.13)$$

$$\|R(u) - R(v)\|_{L^{\frac{2d}{d-2}}(\mathbb{R}^d)} \lesssim \left(\|u\|_{L^{\frac{2d}{d-2}}} + \|v\|_{L^{\frac{2d}{d-2}}} + \|u\|_{L^{\frac{2d}{d-2}}}^2 + \|v\|_{L^{\frac{2d}{d-2}}}^2 \right) \|u - v\|_{L^{\frac{2d}{d-2}}}. \quad (2.14)$$

Proof. By Lemma 2.2 and Hölder's inequality, we have

$$\|\nabla(Vh)\|_{N(I)} \lesssim \|W\|_{Z(I)}^2 \|\nabla h\|_{S(I)} + \|W\|_{Z(I)} \|\nabla W\|_{S(I)} \|h\|_{Z(I)}.$$

Since $W \in L_x^{\frac{6d}{3d-8}} \cap \dot{W}_x^{1, \frac{6d}{3d-4}}$, we have that

$$\|W\|_{Z(I)} \lesssim |I|^{\frac{1}{6}}, \quad \|\nabla W\|_{S(I)} \lesssim |I|^{\frac{1}{3}}.$$

This implies (2.12). The analogue argument gives that

$$\begin{aligned} & \|\nabla(R(g) - R(h))\|_{N(I)} \\ & \lesssim \|g - h\|_{Z(I)} \left(\|\nabla(g + h)\|_{S(I)} |I|^{\frac{1}{6}} + \|g + h\|_{Z(I)} |I|^{\frac{1}{3}} \right) \\ & \quad + \|\nabla(g - h)\|_{S(I)} \|g + h\|_{Z(I)} |I|^{\frac{1}{6}} \\ & \quad + \|g - h\|_{Z(I)} \left(\|\nabla g\|_{S(I)} \|g\|_{Z(I)} + \|\nabla(g + h)\|_{S(I)} \|h\|_{Z(I)} \right) \\ & \quad + \|\nabla(g - h)\|_{S(I)} \left(\|g\|_{Z(I)}^2 + \|g + h\|_{Z(I)} \|h\|_{Z(I)} \right) \\ & \quad + \|g - h\|_{Z(I)} \left(|I|^{\frac{1}{3}} (\|g\|_{Z(I)} + \|h\|_{Z(I)}) + |I|^{\frac{1}{6}} (\|\nabla g\|_{S(I)} + \|\nabla h\|_{S(I)}) \right) \\ & \quad + \|\nabla(g - h)\|_{S(I)} \|g\|_{Z(I)} |I|^{\frac{1}{6}} \\ & \lesssim \|g - h\|_{l(I)} \left((|I|^{\frac{1}{6}} + |I|^{\frac{1}{3}}) (\|g\|_{l(I)} + \|h\|_{l(I)}) + \|g\|_{l(I)}^2 + \|h\|_{l(I)}^2 \right). \end{aligned}$$

In addition, (2.14) holds by Lemma 2.2. This completes the proof. \square

Due to the emergence of the linear operator V in the linearized equation (2.9), we will often use the following integral summation argument.

Lemma 2.11 ([16]). *Let $t_0 > 0$, $p \in [1, +\infty[$, $a_0 \neq 0$, E a normed vector space, and $f \in L_{\text{loc}}^p(t_0, +\infty; E)$ such that*

$$\exists \tau_0 > 0, \exists C_0 > 0, \forall t \geq t_0, \quad \|f\|_{L^p(t, t+\tau_0, E)} \leq C_0 e^{a_0 t}. \quad (2.15)$$

Then for $t \geq t_0$,

$$\|f\|_{L^p(t, +\infty, E)} \leq \frac{C_0 e^{a_0 t}}{1 - e^{a_0 \tau_0}}, \quad \text{if } a_0 < 0; \quad (2.16)$$

$$\|f\|_{L^p(t_0, t, E)} \leq \frac{C_0 e^{a_0 t}}{1 - e^{-a_0 \tau_0}}, \quad \text{if } a_0 > 0. \quad (2.17)$$

By the Strichartz estimate, Lemma 2.10 and Lemma 2.11, we have

Lemma 2.12. *Let v be a solution of (2.9) with*

$$\|v(t)\|_{\dot{H}^1} \leq C e^{-c_1 t}$$

for some C and $c_1 > 0$, then for any admissible pair (q, r) , i.e.

$$\frac{2}{q} = d \left(\frac{1}{2} - \frac{1}{r} \right), \quad q \in [2, +\infty],$$

we have for large t

$$\|v\|_{l(t, +\infty)} + \|\nabla v\|_{L^q(t, +\infty; L^r)} \leq C e^{-c_1 t}.$$

Proof. For small τ_0 , by the Strichartz estimate and Lemma 2.10, we have on $I = [t, t + \tau_0]$

$$\begin{aligned} \|v\|_{l(I)} + \|\nabla v\|_{L^q(I; L^r)} & \leq C e^{-c_1 t} + C \|\nabla(Vh)\|_{N(I)} + C \|\nabla R(h)\|_{N(I)} \\ & \leq C e^{-c_1 t} + C |I|^{\frac{1}{3}} \|h\|_{l(I)} + \|h\|_{l(I)}^2 + \|h\|_{l(I)}^3. \end{aligned}$$

By choosing sufficiently small τ_0 , the continuous argument gives that

$$\|v\|_{l(I)} + \|\nabla v\|_{L^q(I; L^r)} \leq C e^{-c_1 t}.$$

This implies the desired result by Lemma 2.11. \square

2.6. Spectral properties of the linearized operator. Due to the symmetries of (1.1) under the phase rotation and the scaling, we know that the elements

$$iW \in L^2(\mathbb{R}^d), \quad \widetilde{W} = \frac{d-2}{2}W + x \cdot \nabla W \in L^2(\mathbb{R}^d)$$

belongs to the null-space of \mathcal{L} in L_{rad}^2 . Indeed, they are the only elements of the null-space of \mathcal{L} in L_{rad}^2 .

Lemma 2.13. *Let \mathcal{L} be defined by (2.9). Then*

$$\{u \in L_{rad}^2(\mathbb{R}^d), \mathcal{L}u = 0\} = \text{span}\{iW, \widetilde{W}\}.$$

Namely,

$$\{u \in L_{rad}^2(\mathbb{R}^d), L_- u = 0\} = \text{span}\{W\}; \quad \{u \in L_{rad}^2(\mathbb{R}^d), L_+ u = 0\} = \text{span}\{\widetilde{W}\}.$$

Proof. We prove it by the oscillation properties of Sturm-Liouville eigenvalue problems. Note that $L_- W = 0$ and the ground state W is non-degenerate (Lemma 2.5), we know that $\{u \in L_{rad}^2, L_- u = 0\} = \text{span}\{W\}$. Hence it suffices to show that

$$\{u \in L_{rad}^2, L_+ u = 0\} = \text{span}\{\widetilde{W}\}.$$

For the radial function, by the spherical coordinates (see [26, 64]), $L_+ v = 0$ can be written as

$$\begin{aligned} A_0 v(r) = & -\frac{d^2}{dr^2} v - \frac{d-1}{r} \frac{d}{dr} v - \int_0^\infty \rho^{d-5} V\left(\frac{r}{\rho}\right) W(\rho)^2 d\rho v(r) \\ & - 2 \int_0^\infty \rho^{d-5} V\left(\frac{r}{\rho}\right) W(\rho) v(\rho) d\rho W(r), \end{aligned}$$

where

$$V(\rho) = \int_{\mathbb{S}^{d-1}} \frac{1}{|\rho - y'|^4} d\sigma_{y'} = \omega_{d-2} \int_{-1}^1 \frac{(1-s^2)^{\frac{d-3}{2}}}{(\rho^2 - 2\rho s + 1)^2} ds,$$

and \mathbb{S}^{d-1} denotes the unit sphere in \mathbb{R}^d and ω_{d-2} denotes the area of the unit sphere \mathbb{S}^{d-2} in \mathbb{R}^{d-1} .

From $(W, A_0 W) < 0$, we know that the first eigenvalue is negative. By the nonnegative of L_+ on $\{\Delta W\}^\perp$ in Step 1 of Appendix B and the Courant's min-max principle [47], we conclude that the second eigenvalue is nonnegative. Note that $A_0 \widetilde{W}(r) = 0$ and $\widetilde{W} \in L^2$ have only one positive zero, we know that 0 is the second eigenvalue, and by the Sturm-Liouville theory, we conclude the desired result. \square

Now we define the linearized energy Φ by

$$\begin{aligned} \Phi(h) &:= \frac{1}{2} \int_{\mathbb{R}^d} (L_+ h_1) h_1 dx + \frac{1}{2} \int_{\mathbb{R}^d} (L_- h_2) h_2 dx, \\ &= \frac{1}{2} \int_{\mathbb{R}^d} |\nabla h_1|^2 - \frac{1}{2} \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|W(x)|^2 |h_1(y)|^2}{|x-y|^4} - \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{W(x) h_1(x) W(y) h_1(y)}{|x-y|^4} \\ &\quad + \frac{1}{2} \int_{\mathbb{R}^d} |\nabla h_2|^2 - \frac{1}{2} \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|W(x)|^2 |h_2(y)|^2}{|x-y|^4}. \end{aligned} \tag{2.18}$$

We denote by $B(g, h)$ the bilinear symmetric form associated to Φ ,

$$B(g, h) := \frac{1}{2} \int_{\mathbb{R}^d} (L_+ g_1) h_1 \, dx + \frac{1}{2} \int_{\mathbb{R}^d} (L_- g_2) h_2 \, dx, \quad \forall g, h \in \dot{H}^1(\mathbb{R}^d). \quad (2.19)$$

Let us specify the important coercivity properties of Φ . Consider the three orthogonal directions W, iW , and \widetilde{W} in the Hilbert space $\dot{H}^1(\mathbb{R}^d, \mathbb{C})$. Let $H := \text{span}\{W, iW, \widetilde{W}\}$, and

$$H^\perp := \left\{ v \in \dot{H}^1, (iW, v)_{\dot{H}^1} = (\widetilde{W}, v)_{\dot{H}^1} = (W, v)_{\dot{H}^1} = 0 \right\}$$

denotes the orthogonal subspace of H in \dot{H}^1 .

Proposition 2.14. *There exists a constant $c > 0$ such that for all radial function $h \in H^\perp$, we have*

$$\Phi(h) \geq c \|h\|_{\dot{H}^1}^2. \quad (2.20)$$

Proposition 2.15. *Let $\sigma(\mathcal{L})$ be the spectrum of the operator \mathcal{L} , defined on $L^2(\mathbb{R}^d)$ with domain H^2 and let $\sigma_{\text{ess}}(\mathcal{L})$ be its essential spectrum. Then we have*

- (a) *The operator \mathcal{L} admits two radial eigenfunctions $\mathcal{Y}_\pm \in \mathcal{S}(\mathbb{R}^d)$ with real eigenvalues $\pm e_0$, $e_0 > 0$, that is, $\mathcal{L}\mathcal{Y}_\pm = \pm e_0 \mathcal{Y}_\pm$, $\mathcal{Y}_+ = \overline{\mathcal{Y}_-}$, $e_0 > 0$.*
- (b) *There exists a constant $c > 0$ such that for all radial function $h \in G_\perp$, we have*

$$\Phi(h) \geq c \|h\|_{\dot{H}^1}^2,$$

where

$$G_\perp = \left\{ v \in \dot{H}^1, (iW, v)_{\dot{H}^1} = (\widetilde{W}, v)_{\dot{H}^1} = B(\mathcal{Y}_\pm, v) = 0 \right\}.$$

$$(c) \quad \sigma_{\text{ess}}(\mathcal{L}) = \{i\xi : \xi \in \mathbb{R}\}, \quad \sigma(\mathcal{L}) \cap \mathbb{R} = \{-e_0, 0, e_0\}.$$

The proof of Proposition 2.14 and Proposition 2.15 are analogue to that of Claim 3.5, Lemma 5.1 and Corollary 5.3 in [16]. For the sake of completeness, We prove it in Appendix B and Appendix C.

Remark 2.16. As a consequence of the definition of Φ and B , we have for any $f, g \in \dot{H}^1, \mathcal{L}f, \mathcal{L}g \in \dot{H}^1$, and $h \in H^\perp$

$$\Phi(iW) = \Phi(\widetilde{W}) = \Phi(\mathcal{Y}_\pm) = 0, \quad \Phi(W) = -\|\nabla W\|_2^2 < 0, \quad (2.21)$$

$$B(f, g) = B(g, f), \quad B(\mathcal{L}f, g) = -B(f, \mathcal{L}g), \quad (2.22)$$

$$B(iW, f) = B(\widetilde{W}, f) = 0, \quad B(W, h) = 0. \quad (2.23)$$

Corollary 2.17. *As a consequence of Proposition 2.14, we have*

$$B(\mathcal{Y}_+, \mathcal{Y}_-) \neq 0. \quad (2.24)$$

Proof. If $B(\mathcal{Y}_+, \mathcal{Y}_-) = 0$, then for any $h \in \text{span}\{iW, \widetilde{W}, \mathcal{Y}_\pm\}$, which is of dimension 4, we have

$$\Phi(h) = 0.$$

But by Proposition 2.14, we know that Φ is positive on $H^\perp \cap \dot{H}_{\text{rad}}^1$, which is of codimension 3. It is a contradiction. \square

Remark 2.18. As a consequence of Proposition 2.15, we have Lemma 2.13. In fact, it suffices to show the inclusion relation “ \subseteq ”. If the dimension of $\{u \in L_{\text{rad}}^2, \mathcal{L}u = 0\}$ was strictly higher than two, we could find $\mathcal{Z} \in L_{\text{rad}}^2$, $\mathcal{Z} \neq 0$, such that $\mathcal{L}\mathcal{Z} = 0$. Let

$$\mathcal{Z}' = \mathcal{Z} - (\mathcal{Z}, iW)_{\dot{H}^1} \frac{iW}{\|W\|_{\dot{H}^1}^2} - (\mathcal{Z}, \widetilde{W})_{\dot{H}^1} \frac{\widetilde{W}}{\|\widetilde{W}\|_{\dot{H}^1}^2} \neq 0,$$

which implies that $\mathcal{L}\mathcal{Z}' = 0$ and

$$(\mathcal{Z}', iW)_{\dot{H}^1} = 0, (\mathcal{Z}', \widetilde{W})_{\dot{H}^1} = 0, B(\mathcal{Y}_\pm, \mathcal{Z}') = \pm \frac{1}{e_0} B(\mathcal{L}\mathcal{Y}_\pm, \mathcal{Z}') = \mp B(\mathcal{Y}_\pm, \mathcal{L}\mathcal{Z}') = 0.$$

Thus $\mathcal{Z}' \in G_\perp \setminus \{0\}$. While $\Phi(\mathcal{Z}') = (\mathcal{L}\mathcal{Z}', \mathcal{Z}')_{L^2} = 0$, which contradicts the coercivity of Φ on $G_\perp \cap \dot{H}_{rad}^1$.

Corollary 2.19. *As a consequence of Proposition 2.15, we have*

$$\int \nabla W \cdot \nabla \mathcal{Y}_1 \, dx \neq 0, \quad \text{where } \mathcal{Y}_1 = \Re \mathcal{Y}_+. \quad (2.25)$$

Proof. On one hand, we have

$$(W, iW)_{\dot{H}^1} = (W, \widetilde{W})_{\dot{H}^1} = 0.$$

On the other hand, by (2.2), we know $L_+(W) = 2\Delta W$. If $(W, \mathcal{Y}_1)_{\dot{H}^1} = 0$, then by (2.22), we obtain

$$\begin{aligned} e_0 B(\mathcal{Y}_\pm, W) &= B(\mathcal{L}\mathcal{Y}_\pm, W) = -B(\mathcal{Y}_\pm, \mathcal{L}W) \\ &= -\frac{1}{2} \int L_- \mathcal{Y}_2 \cdot L_+(W) = \frac{1}{2} \int e_0 \mathcal{Y}_1 \cdot L_+(W) = \int e_0 \mathcal{Y}_1 \cdot \Delta W = 0, \end{aligned}$$

which means that $W \in G_\perp$. By Proposition 2.15, we have $\Phi(W) \gtrsim \|W\|_{\dot{H}^1}^2$. This contradicts the fact (2.21). This completes the proof. \square

3. EXISTENCE OF SPECIAL SOLUTIONS

In this section, we first construct a family of approximate solutions to (1.1) by making use of the knowledge about the real eigenvalues of the linearized operator \mathcal{L} , and then prove the existence of U^a by a fixed point argument around an approximate solution.

3.1. A family of approximate solutions converging to W as $t \rightarrow +\infty$.

Proposition 3.1. *Let $a \in \mathbb{R}$. There exists a sequence of functions $(\mathcal{Z}_j^a)_{j \geq 1}$ in $\mathcal{S}(\mathbb{R}^d)$ such that $\mathcal{Z}_1^a = a\mathcal{Y}_+$, and if*

$$h_k^a(t, x) := \sum_{j=1}^k e^{-je_0 t} \mathcal{Z}_j^a, \quad k \in \mathbb{Z}^+, \quad (3.1)$$

then as $t \rightarrow +\infty$,

$$\epsilon_k := \partial_t h_k^a + \mathcal{L}h_k^a - R(h_k^a) = O(e^{-(k+1)e_0 t}) \quad \text{in } \mathcal{S}(\mathbb{R}^d). \quad (3.2)$$

Remark 3.2. Let

$$U_k^a(t, x) := W(x) + h_k^a(t, x), \quad (3.3)$$

then as $t \rightarrow +\infty$,

$$\epsilon_k := i\partial_t U_k^a + \Delta U_k^a + \left(\frac{1}{|x|^4} * |U_k^a|^2 \right) U_k^a = O(e^{-(k+1)e_0 t}) \quad \text{in } \mathcal{S}(\mathbb{R}^d).$$

Proof of Proposition 3.1. We prove it by induction. Note that $h_1^a := e^{-e_0 t} \mathcal{Z}_1^a$, we have

$$\partial_t h_1^a + \mathcal{L}h_1^a - R(h_1^a) = -R(h_1^a) = -R(e^{-e_0 t} \mathcal{Z}_1^a).$$

By the definition of (2.8), we know that R has at least the quadratic term, this implies that $R(e^{-e_0 t} \mathcal{Z}_1^a) = O(e^{-2e_0 t})$. Therefore, we conclude (3.2) for $k = 1$.

Let $k \geq 1$ and assume that there exist $\mathcal{Z}_1^a, \dots, \mathcal{Z}_k^a$ such that h_k^a satisfies (3.2), then there exists $\mathcal{U}_{k+1}^a \in \mathcal{S}$ such that, as $t \rightarrow +\infty$,

$$\partial_t h_k^a + \mathcal{L}h_k^a = R(h_k^a) + e^{-(k+1)e_0 t} \mathcal{U}_{k+1}^a + O\left(e^{-(k+2)e_0 t}\right) \quad \text{in } \mathcal{S}.$$

By Proposition 2.15, $(k+1)e_0$ is not in the spectrum of \mathcal{L} . Define

$$\mathcal{Z}_{k+1}^a := -(\mathcal{L} - (k+1)e_0)^{-1} \mathcal{U}_{k+1}^a.$$

By the similar argument as in the proof in [16, Remark 7.2], we know that $\mathcal{Z}_{k+1}^a \in \mathcal{S}$. Then we have

$$\partial_t \left(h_k^a + e^{-(k+1)e_0 t} \mathcal{Z}_{k+1}^a \right) + \mathcal{L} \left(h_k^a + e^{-(k+1)e_0 t} \mathcal{Z}_{k+1}^a \right) = R(h_k^a) + O\left(e^{-(k+2)e_0 t}\right). \quad (3.4)$$

Denote

$$h_{k+1}^a := h_k^a + e^{-(k+1)e_0 t} \mathcal{Z}_{k+1}^a.$$

By (3.4), h_{k+1}^a satisfies

$$\partial_t h_{k+1}^a + \mathcal{L}h_{k+1}^a - R(h_{k+1}^a) = R(h_k^a) - R(h_{k+1}^a) + O\left(e^{-(k+2)e_0 t}\right) \quad \text{as } t \rightarrow +\infty. \quad (3.5)$$

Since

$$h_j^a = O\left(e^{-e_0 t}\right) \quad \text{for } j = k, k+1, \quad \text{and } h_k^a - h_{k+1}^a = O\left(e^{-(k+1)e_0 t}\right) \quad \text{as } t \rightarrow +\infty,$$

we obtain

$$R(h_k^a) - R(h_{k+1}^a) = O\left(e^{-(k+2)e_0 t}\right) \quad \text{as } t \rightarrow +\infty.$$

This together with (3.5) shows the desired estimate (3.2) for $k+1$, so we completes the proof. \square

3.2. Construction of special solutions near an approximate solution. Now we prove the existence of the special solution with the threshold energy by a fixed point argument.

Proposition 3.3. *Let $a \in \mathbb{R}$. There exist $k_0 > 0$ and $t_0 \geq 0$ such that for any $k \geq k_0$, there exists a radial solution U^a of (1.1) such that for $t \geq t_0$*

$$\|U^a - U_k^a\|_{l(t, +\infty)} \leq e^{-(k+\frac{1}{2})e_0 t}. \quad (3.6)$$

Furthermore U^a is the unique solution of (1.1) satisfying (3.6) for large t . Finally, U^a is independent of k and satisfies, for $t \geq t_0$,

$$\|U^a(t) - W - ae^{-e_0 t} \mathcal{Y}_+\|_{\dot{H}^1} \leq e^{-\frac{3}{2}e_0 t}. \quad (3.7)$$

Proof. The function U^a is solution of (1.1) if and only if $h^a := U^a - W$ is solution of

$$\partial_t h^a + \mathcal{L}h^a = R(h^a).$$

By (3.2), $h_k^a := U_k^a - W$ satisfies that

$$\partial_t h_k^a + \mathcal{L}h_k^a = R(h_k^a) + \epsilon_k.$$

Therefore, U^a satisfies (1.1) if and only if $e := U^a - U_k^a = h^a - h_k^a$ satisfies

$$\partial_t e + \mathcal{L}e = R(h_k^a + e) - R(h_k^a) - \epsilon_k.$$

This may be rewritten

$$i\partial_t e + \Delta e = V e + iR(h_k^a + e) - iR(h_k^a) - i\epsilon_k. \quad (3.8)$$

Let

$$\mathcal{M}_k(e)(t) := - \int_t^{+\infty} e^{i(t-s)\Delta} \left(iVe(s) - R(h_k^a(s) + e(s)) + R(h_k^a(s)) + \varepsilon_k(s) \right) ds.$$

It is easy to see that the solution U^a of (1.1) satisfying (3.6) for $t \geq t_0$ is equivalent to the fixed point of the following integral equation

$$\forall t \geq t_0, \quad e(t) = \mathcal{M}_k(e)(t) \quad \text{and} \quad \|e\|_{l(t,+\infty)} \leq e^{-(k+\frac{1}{2})e_0 t}. \quad (3.9)$$

Let us fix k and t_0 . Denote

$$E_l^k := \left\{ e \in Z(t_0, +\infty), \nabla e \in S(t_0, +\infty); \|e\|_{E_l^k} := \sup_{t \geq t_0} e^{(k+\frac{1}{2})e_0 t} \|e\|_{l(t,+\infty)} < \infty \right\},$$

$$B_l^k := \left\{ e \in Z(t_0, +\infty), \nabla e \in S(t_0, +\infty); \sup_{t \geq t_0} e^{(k+\frac{1}{2})e_0 t} \|e\|_{l(t,+\infty)} \leq 1 \right\}.$$

The space E_l^k is a Banach space. In view of (3.9), it is sufficient to show that if t_0 and k are large enough, the mapping \mathcal{M}_k is a contraction on B_l^k .

Let $e \in B_l^k$, $k \geq 1$. By the Strichartz estimate, we have

$$\|\mathcal{M}_k(e)\|_{l(t,+\infty)} \leq C^* \left(\|\nabla(Ve)\|_{N(t,+\infty)} + \|\nabla(R(h_k^a + e) - R(h_k^a))\|_{N(t,+\infty)} + \|\nabla\varepsilon_k\|_{N(t,+\infty)} \right).$$

We first estimate $\|\nabla(Ve)\|_{N(t,+\infty)}$. Let $\tau_0 \in (0, 1)$. By Lemma 2.10, we have

$$\|\nabla(Ve)\|_{N(t,t+\tau_0)} \leq C\tau_0^{\frac{1}{3}} \|e\|_{l(t,t+\tau_0)} \leq C\tau_0^{\frac{1}{3}} e^{-(k+\frac{1}{2})e_0 t} \|e\|_{E_l^k} \leq C\tau_0^{\frac{1}{3}} e^{-(k+\frac{1}{2})e_0 t}.$$

This together with Lemma 2.11 yields that

$$\|\nabla(Ve)\|_{N(t,+\infty)} \leq \frac{C\tau_0^{\frac{1}{3}}}{1 - e^{-(k+\frac{1}{2})e_0 \tau_0}} e^{-(k+\frac{1}{2})e_0 t}. \quad (3.10)$$

To estimate $\|\nabla(R(h_k^a + e) - R(h_k^a))\|_{N(t,+\infty)}$. By Lemma 2.10, we have

$$\begin{aligned} & \|\nabla(R(h_k^a + e) - R(h_k^a))\|_{N(t,t+1)} \\ & \leq C\|e\|_{l(t,t+1)} \left(\|h_k^a\|_{l(t,t+1)} + \|e\|_{l(t,t+1)} + \|h_k^a\|_{l(t,t+1)}^2 + \|e\|_{l(t,t+1)}^2 \right) \\ & \leq Ce^{-(k+\frac{3}{2})e_0 t} \|e\|_{E_l^k} \leq Ce^{-(k+\frac{3}{2})e_0 t}. \end{aligned}$$

Thus, we obtain by Lemma 2.11

$$\|\nabla(R(h_k^a + e) - R(h_k^a))\|_{N(t,+\infty)} \leq \frac{C}{1 - e^{-(k+\frac{3}{2})e_0}} e^{-(k+\frac{3}{2})e_0 t}. \quad (3.11)$$

Finally, we estimate $\|\nabla\varepsilon_k\|_{N(t,+\infty)}$. By $\varepsilon_k = O(e^{-(k+1)e_0 t})$ for large t , we have

$$\|\nabla\varepsilon_k\|_{N(t,+\infty)} \leq \frac{C}{(k+1)e_0} e^{-(k+1)e_0 t}. \quad (3.12)$$

By (3.10), (3.11) and (3.12), we have

$$\|\mathcal{M}_k(e)\|_{l(t,+\infty)} \leq C^* \left(\frac{C\tau_0^{\frac{1}{3}}}{1 - e^{-(k+\frac{1}{2})e_0 \tau_0}} + \frac{Ce^{-e_0 t}}{1 - e^{-(k+\frac{3}{2})e_0}} + \frac{Ce^{-\frac{1}{2}e_0 t}}{(k+1)e_0} \right) e^{-(k+\frac{1}{2})e_0 t}.$$

If choosing a small τ_0 , and a large k_0 and t_0 such that

$$C^* C \tau_0^{\frac{1}{3}} = \frac{1}{8}, \quad e^{-(k_0 + \frac{1}{2})e_0 \tau_0} \leq \frac{1}{2}, \quad \text{and} \quad C^* \left(\frac{C e^{-e_0 t_0}}{1 - e^{-(k_0 + \frac{3}{2})e_0}} + \frac{C e^{-\frac{1}{2}e_0 t_0}}{(k_0 + 1)e_0} \right) \leq \frac{1}{4},$$

we have

$$\|\mathcal{M}_k(e)\|_{E_t^k} \leq \frac{1}{2}, \quad k \geq k_0, \quad t \geq t_0. \quad (3.13)$$

By the similar analysis, we can obtain for any $e, f \in B_t^k$, $k \geq 1$,

$$\begin{aligned} & \|\mathcal{M}_k(e) - \mathcal{M}_k(f)\|_{l(t, +\infty)} \\ & \leq C^* \left(\|\nabla(Ve - Vf)\|_{N(t, +\infty)} + \|\nabla(R(h_k^a + e) - R(h_k^a + f))\|_{N(t, +\infty)} \right) \\ & \leq C^* \left(\frac{C \tau_0^{\frac{1}{3}}}{1 - e^{-(k + \frac{1}{2})e_0 \tau_0}} + \frac{C e^{-e_0 t}}{1 - e^{-(k + \frac{3}{2})e_0}} \right) e^{-(k + \frac{1}{2})e_0 t} \|e - f\|_{E_t^k} \\ & \leq \frac{1}{2} e^{-(k + \frac{1}{2})e_0 t} \|e - f\|_{E_t^k}. \end{aligned} \quad (3.14)$$

For $k \geq k_0$, and large t_0 , (3.13) and (3.14) show that \mathcal{M}_k is a contraction map on B_t^k . Thus for $k \geq k_0$, (1.1) has an unique solution U^a satisfying (3.6) for $t \geq t_0$.

Next we show that U^a does not depend on k . Since the above proceeding still remains valid for the larger t_0 , the uniqueness still holds in the class of solution of (1.1) satisfying (3.6) for $t \geq t'_0$, where t'_0 is any real number larger than t_0 . Let $k < \tilde{k}$, and U^a and \tilde{U}^a be the solutions of (1.1) constructed for k and \tilde{k} respectively. The uniqueness of the fixed point show that $U^a = \tilde{U}^a$ for large t , then by the uniqueness of (1.1), we have $U^a = \tilde{U}^a$.

Finally, we prove (3.7). Note that

$$U^a - U_k^a = e = \mathcal{M}_k(e).$$

By the Strichartz estimate, we have

$$\begin{aligned} & \|U^a - U_k^a\|_{\dot{H}^1} = \|e\|_{\dot{H}^1} \\ & \leq C^* \left(\|\nabla(Ve)\|_{N(t, +\infty)} + \|\nabla(R(h_k^a + e) - R(h_k^a))\|_{N(t, +\infty)} + \|\nabla \varepsilon_k\|_{N(t, +\infty)} \right) \\ & \leq e^{-(k + \frac{1}{2})e_0 t}. \end{aligned}$$

This together with the fact $U_k^a = W + a e^{-e_0 t} \mathcal{Y}_+ + O(e^{-2e_0 t})$ in \dot{H}^1 yields (3.7). \square

3.3. Construction of W^\pm . Note that (3.7) and the conservation of energy, we have

$$E(U^a) = E(W).$$

In addition, we have by (3.7)

$$\|\nabla U^a(t)\|_2^2 = \|\nabla W\|_2^2 + 2a e^{-e_0 t} \int_{\mathbb{R}^d} \nabla W \cdot \nabla \mathcal{Y}_1 \, dx + O\left(e^{-\frac{3}{2}e_0 t}\right) \quad \text{as } t \rightarrow +\infty.$$

By (2.25), replacing \mathcal{Y}_+ by $-\mathcal{Y}_+$ if necessary, we may assume that

$$\int \nabla W \cdot \nabla \mathcal{Y}_1 \, dx > 0.$$

This implies that $\|\nabla U^a(t)\|_2^2 - \|\nabla W\|_2^2$ enjoys the same sign with a for large positive time. Thus, by Lemma 2.8, $\|\nabla U^a(t_0)\|_2^2 - \|\nabla W\|_2^2$ has the same sign to that of a .

Let

$$W^+(t, x) = U^{+1}(t + t_0, x), \quad W^-(t, x) = U^{-1}(t + t_0, x), \quad (3.15)$$

which yields two radial solutions $W^\pm(t, x)$ of (1.1) for $t \geq 0$, and satisfies

$$E(W^\pm(t)) = E(W), \quad \|\nabla W^-(0)\|_2 < \|\nabla W\|_2, \quad \|\nabla W^+(0)\|_2 > \|\nabla W\|_2,$$

and

$$\|W^\pm(t) - W\|_{\dot{H}^1} \leq Ce^{-e_0 t}, \quad t \geq 0.$$

To complete the proof of Theorem 1.2, it remains to show the behavior of W^\pm for the negative time, we will do it in Section 5.3 and Section 6.6. \square

4. MODULATION

For the radial function $u \in \dot{H}^1(\mathbb{R}^d)$, we define the gradient variant of the solution away from W as

$$\delta(u) = \left| \int_{\mathbb{R}^d} (|\nabla u(x)|^2 - |\nabla W(x)|^2) dx \right|.$$

By Proposition 2.7, we know that if

$$E(u) = E(W) \quad (4.1)$$

and $\delta(u)$ is small enough, then there exists $\tilde{\theta}$ and $\tilde{\mu}$ such that

$$u_{\tilde{\theta}, \tilde{\mu}} = W + \tilde{u}, \quad \text{with } \|\tilde{u}\|_{\dot{H}^1} \leq \varepsilon(\delta(u)),$$

where $\varepsilon(\delta) \rightarrow 0$ as $\delta \rightarrow 0$. The goal in this section is that for the solution of (1.1) with small gradient variant, we choose parameters $\tilde{\theta}$ and $\tilde{\mu}$ to show the linear dependence between these parameters, their derivatives and $\delta(u)$.

From the implicit theorem and the variational characterization of W in Proposition 2.7, we have the following orthogonal decomposition.

Lemma 4.1. *There exist $\delta_0 > 0$ and a positive function $\epsilon(\delta)$ defined for $0 < \delta \leq \delta_0$, which tends to 0 when δ tends to 0, such that for all radial u in $\dot{H}^1(\mathbb{R}^d)$ satisfying (4.1), there exists a couple $(\theta, \mu) \in \mathbb{R} \times (0, +\infty)$ such that $v = u_{\theta, \mu}$ satisfies*

$$v \perp iW, \quad v \perp \widetilde{W}. \quad (4.2)$$

The parameters (θ, μ) are unique in $\mathbb{R}/2\pi\mathbb{Z} \times \mathbb{R}$, and the mapping $u \mapsto (\theta, \mu)$ is C^1 .

Proof. From Proposition 2.7, there exist a function ϵ and $\theta_1, \mu_1 > 0$ such that $u_{\theta_1, \mu_1} = W + g$ and

$$\|u_{\theta_1, \mu_1} - W\|_{\dot{H}^1} \leq \epsilon(\delta(u)). \quad (4.3)$$

Now we consider the functional

$$\begin{aligned} J(\theta, \mu, f) &= (J_0(\theta, \mu, f), J_1(\theta, \mu, f)) = \left((f_{\theta, \mu}, iW)_{\dot{H}^1}, (f_{\theta, \mu}, \widetilde{W})_{\dot{H}^1} \right) \\ &= \left((e^{i\theta} \mu^{-\frac{d-2}{2}} f(x/\mu), iW)_{\dot{H}^1}, (e^{i\theta} \mu^{-\frac{d-2}{2}} f(x/\mu), \widetilde{W})_{\dot{H}^1} \right). \end{aligned}$$

By the simple computations, we have

$$J(0, 1, W) = 0, \quad \left| \frac{\partial J}{\partial(\theta, \mu)}(0, 1, W) \right| = \begin{vmatrix} \|\nabla W\|_{L^2}^2 & 0 \\ 0 & -\|\nabla \widetilde{W}\|_{L^2}^2 \end{vmatrix} \neq 0.$$

Thus by the implicit theorem, there exist $\epsilon_0, \eta_0 > 0$, such that for any $h \in \dot{H}^1$ with $\|h - W\|_{\dot{H}^1} < \epsilon_0$, there exists a unique $(\tilde{\theta}(h), \tilde{\mu}(h)) \in C^1$ satisfying $|\tilde{\theta}| + |\tilde{\mu} - 1| < \eta_0$ and

$$J(\tilde{\theta}, \tilde{\mu}, h) = 0.$$

By (4.3), this implies that there exists a unique $(\tilde{\theta}_1(u), \tilde{\mu}_1(u))$ such that

$$J(\tilde{\theta}_1 + \theta_1, \tilde{\mu}_1 \mu_1, u) = J(\tilde{\theta}_1, \tilde{\mu}_1, u_{\theta_1, \mu_1}) = 0.$$

This completes the proof by taking $\theta = \tilde{\theta}_1 + \theta_1$ and $\mu = \tilde{\mu}_1 \mu_1$. \square

Let $u(t)$ be a radial solution of (1.1) satisfying (4.1) and write

$$\delta(t) := \delta(u(t)) = \left| \int_{\mathbb{R}^d} (|\nabla u(t, x)|^2 - |\nabla W(x)|^2) dx \right|. \quad (4.4)$$

Let D_{δ_0} be the open set of all times t in the domain of existence of u such that $\delta(t) < \delta_0$. On D_{δ_0} , by Lemma 4.1, there exists C^1 functions $\theta(t)$ and $\mu(t)$ with the following decomposition

$$u_{\theta(t), \mu(t)}(t, x) = (1 + \alpha(t))W(x) + h(t, x), \quad (4.5)$$

$$1 + \alpha(t) = \frac{1}{\|W\|_{\dot{H}^1}^2} (u_{\theta(t), \mu(t)}, W)_{\dot{H}^1}, \quad h \in H^\perp \cap \dot{H}_{rad}^1.$$

In Section 5 and Section 6, we will make use of additional conditions to show that u converges exponentially to W in \dot{H}^1 , up to the constant modulation parameters.

As a consequence of the coercivity of Φ on $H^\perp \cap \dot{H}_{rad}^1$ in Proposition 2.14, we can identify the scaling and phase parameters on the interval with small gradient variant $\delta(t)$, which can be controlled by the gradient variant.

Lemma 4.2. *Let u be a radial solution of (1.1) satisfying (4.1). Then taking a smaller δ_0 if necessary, the following estimates hold for $t \in D_{\delta_0}$:*

$$|\alpha(t)| \approx \|\alpha(t)W(\cdot) + h(t, \cdot)\|_{\dot{H}^1} \approx \|h(t, \cdot)\|_{\dot{H}^1} \approx \delta(t), \quad (4.6)$$

$$|\alpha'(t)| + |\theta'(t)| + \left| \frac{\mu'(t)}{\mu(t)} \right| \leq C\mu^2(t)\delta(t). \quad (4.7)$$

Proof. Denote

$$\tilde{\delta}(t) := |\alpha(t)| + \|\alpha(t)W + h(t)\|_{\dot{H}^1} + \|h(t)\|_{\dot{H}^1} + \delta(t).$$

We first show that

$$\tilde{\delta}(t) \lesssim |\alpha(t)|. \quad (4.8)$$

Indeed, by (4.5), $E(u) = E(W)$ and $t \in D_{\delta_0}$, we have

$$\Phi(\alpha(t)W + h(t)) = O\left(\|\nabla(\alpha(t)W + h(t))\|_2^3\right). \quad (4.9)$$

By (2.23), we know that W and $h(t)$ are B -orthogonality, and $B(f, f) = \Phi(f)$, thus

$$\Phi(\alpha(t)W + h(t)) = -\alpha(t)^2|\Phi(W)| + \Phi(h(t)). \quad (4.10)$$

By (4.9), (4.10) and the coercivity of Φ on $H^\perp \cap \dot{H}_{rad}^1$, we have

$$\|h(t)\|_{\dot{H}^1}^2 \approx \Phi(h(t)) \approx \alpha(t)^2 + O\left(\|\nabla(\alpha(t)W + h(t))\|_2^3\right) \lesssim \alpha(t)^2 + \tilde{\delta}^3(t). \quad (4.11)$$

By the orthogonality of W and h in \dot{H}^1 , we have

$$\|\nabla(\alpha(t)W + h(t))\|_2^2 = \alpha(t)^2 \|\nabla W\|_2^2 + \|\nabla h(t)\|_2^2 \lesssim \alpha(t)^2 + \tilde{\delta}^3(t). \quad (4.12)$$

and

$$\delta(t) = \left| \int_{\mathbb{R}^d} |\nabla u(t)|^2 - |\nabla W|^2 dx \right| = \left| (2\alpha(t) + \alpha(t)^2) \|\nabla W\|_2^2 + \|\nabla h(t)\|_2^2 \right|. \quad (4.13)$$

If δ_0 is small, then $\|\nabla(\alpha(t)W + h(t))\|_2$ and $|\alpha(t)|$ are small (by the orthogonality of W and h in \dot{H}^1), thus we have

$$\delta(t) \lesssim |\alpha(t)| + \tilde{\delta}^3(t).$$

Therefore, we obtain that

$$\tilde{\delta}(t) \lesssim |\alpha(t)| + \tilde{\delta}^{\frac{3}{2}}(t) + \tilde{\delta}^3(t).$$

By the continuity argument, we have (4.8). By (4.11), (4.12) and (4.13) once again, we have

$$|\alpha(t)| \approx \|h(t)\|_{\dot{H}^1} \approx \|\alpha W + h\|_{\dot{H}^1} \approx \delta(t).$$

Next we prove (4.7). Denote

$$\tilde{\tilde{\delta}}(t) := |\alpha'(t)| + |\theta'(t)| + \left| \frac{\mu'(t)}{\mu(t)} \right| + \mu^2(t)\delta(t).$$

Let

$$v(t, y) = u_{\theta(t), \mu(t)}(t, y) = e^{i\theta(t)} \mu(t)^{-\frac{d-2}{2}} u\left(t, \frac{y}{\mu(t)}\right),$$

then

$$u(t, x) = e^{-i\theta(t)} \mu^{\frac{d-2}{2}}(t) v(t, \mu(t)x),$$

and (1.1) may be written

$$i\partial_t v + \mu^2 \Delta v + \mu^2 \left(\frac{1}{|x|^4} * |v|^2 \right) v + \theta' v + i \frac{\mu'}{\mu} \left(\frac{d-2}{2} + y \cdot \nabla \right) v = 0.$$

For $t \in D_{\delta_0}$, by (4.5), we have $v = (1 + \alpha)W + h$, $h \in H^\perp$. One easily verifies that h satisfies that

$$i\partial_t h + \mu^2(t)\Delta h + i\alpha'(t)W + \theta'(t)W + i \frac{\mu'(t)}{\mu(t)} \widetilde{W} = O\left(\mu^2(t)\delta(t) + \delta(t)\tilde{\tilde{\delta}}(t)\right) \text{ in } \dot{H}^1.$$

Note that $h \in H^\perp$ for $t \in D_{\delta_0}$, we have $\partial_t h \in H^\perp$, i.e.

$$\Re \int \partial_t h \Delta W = \Im \int \partial_t h \Delta W = \Re \int \partial_t h \Delta \widetilde{W} = 0.$$

This together with (4.6) implies that

$$\begin{aligned} |\alpha'(t)| &= O\left(\mu^2(t)\delta(t) + \delta(t)\tilde{\tilde{\delta}}(t)\right), \quad |\theta'(t)| = O\left(\mu^2(t)\delta(t) + \delta(t)\tilde{\tilde{\delta}}(t)\right), \\ \left| \frac{\mu'(t)}{\mu(t)} \right| &= O\left(\mu^2(t)\delta(t) + \delta(t)\tilde{\tilde{\delta}}(t)\right). \end{aligned}$$

Therefore, we obtain that

$$\tilde{\tilde{\delta}}(t) = O\left(\mu^2(t)\delta(t) + \delta(t)\tilde{\tilde{\delta}}(t)\right).$$

This yields the result if δ_0 is chosen small enough. \square

5. CONVERGENCE TO W FOR THE SUPERCRITICAL THRESHOLD SOLUTION

In this section, we will show the dynamics of W^+ in Theorem 1.2 in the negative time, and it is also the first step in the proof of case (c) of Theorem 1.3.

Proposition 5.1. *Consider a radial solution $u \in H^1(\mathbb{R}^d)$ of (1.1) such that*

$$E(u) = E(W), \quad \|\nabla u_0\|_{L^2} > \|\nabla W\|_{L^2}, \quad (5.1)$$

which is globally defined for the positive times. Then there exist $\theta_0 \in \mathbb{R}/(2\pi\mathbb{Z})$, $\mu_0 \in (0, +\infty)$, $c, C > 0$ such that

$$\forall t \geq 0, \quad \|u - W_{\theta_0, \mu_0}\|_{\dot{H}^1} \leq Ce^{-ct}, \quad (5.2)$$

and the negative time of existence of u is finite.

5.1. Exponential convergence of the gradient variant.

Lemma 5.2. *Under the assumptions of Proposition 5.1, there exists C such that for any $R > 0$, and all $t \geq 0$,*

$$|\partial_t V_R(t)| \leq CR^2 \delta(t), \quad (5.3)$$

where $V_R(t)$ is defined as in (2.6).

Proof. By Lemma 2.9, we have

$$\partial_t V_R(t) = 2\Im \int_{\mathbb{R}^d} \bar{u} \nabla u \cdot \nabla \phi_R \, dx.$$

Note that $|\nabla \phi_R| \lesssim R^2/|x|$, we have by the Hardy inequality

$$|\partial_t V_R(t)| \lesssim R^2 \|u(t)\|_{\dot{H}^1}^2.$$

Thus by Lemma 2.8 and the definition of $\delta(t)$, it suffices to show (5.3) when $\delta(t) \leq \delta_0$, where δ_0 comes from Section 4. In this case, by Lemma 4.2, we can write u as

$$u_{\theta(t), \mu(t)}(t, x) = W + \tilde{u}, \quad \|\tilde{u}\|_{\dot{H}^1} \lesssim \delta(t) \implies u(t, x) = (W + \tilde{u})_{-\theta(t), 1/\mu(t)}.$$

Thus we have

$$\begin{aligned} \partial_t V_R(t) &= 2\Im \int_{\mathbb{R}^d} \overline{(W + \tilde{u})_{-\theta(t), 1/\mu(t)}} \nabla (W + \tilde{u})_{-\theta(t), 1/\mu(t)} \nabla \phi_R \, dx \\ &= \frac{2R}{\mu(t)} \Im \int_{\mathbb{R}^d} \overline{(W + \tilde{u})} \nabla (W + \tilde{u}) \cdot \nabla \phi \left(\frac{x}{R\mu(t)} \right) \, dx \\ &= 2R^2 \Im \int_{\mathbb{R}^d} \frac{1}{R\mu(t)} \left(W \nabla \tilde{u} + \tilde{u} \nabla W + \tilde{u} \nabla \tilde{u} \right) \cdot \nabla \phi \left(\frac{x}{R\mu(t)} \right) \, dx. \end{aligned}$$

By the definition of ϕ and the Hardy inequality, we have

$$|\partial_t V_R(t)| \lesssim R^2 (\|\tilde{u}\|_{\dot{H}^1} + \|\tilde{u}\|_{\dot{H}^1}^2) \lesssim R^2 \delta(t).$$

This completes the proof. \square

Lemma 5.3. *Under the assumptions of Proposition 5.1, there exist $C > 0$ and $R_1 \geq 1$ such that for $R \geq R_1$, and all $t \geq 0$,*

$$\partial_t^2 V_R(t) \leq -4\delta(t), \quad (5.4)$$

$$\partial_t V_R(t) > 0. \quad (5.5)$$

Proof. By Lemma 2.9, we have

$$\partial_t^2 V_R(t) = 8 \int_{\mathbb{R}^d} |\nabla u(t, x)|^2 dx - 8 \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|u(t, x)|^2 |u(t, y)|^2}{|x - y|^4} dx dy + A_R(u(t)),$$

where $A_R(u(t))$ is defined in Lemma 2.9.

By $E(u) = E(W) = \|\nabla W\|_{L^2}^2/4$, we have

$$8 \int_{\mathbb{R}^d} |\nabla u|^2 dx - 8 \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|u(t, x)|^2 |u(t, y)|^2}{|x - y|^4} dx dy = -8\delta(t).$$

To prove (5.4), it suffices to show that for $R \geq R_1$

$$A_R(u(t)) \leq 4\delta(t).$$

We divide it into three steps:

Step 1: General bound on $A_R(u(t))$. We claim that there exists $C > 0$ such that for any $R > 0$, $t \geq 0$,

$$A_R(u(t)) \leq \frac{C}{R^2} + \frac{C}{R^{\frac{4d-4}{d}}} \|u(t)\|_{\dot{H}^1}^{\frac{4}{d}} + \frac{C}{R^2} \|u(t)\|_{\dot{H}^1}^2. \quad (5.6)$$

Indeed, choosing suitably ϕ such that

$$\phi''(r) \leq 2, \quad r \geq 0, \quad \text{and} \quad |-\Delta \Delta \phi_R| \lesssim \frac{1}{R^2},$$

we conclude by the definition $A_R(u)$ in Lemma 2.9

$$\begin{aligned} A_R(u(t)) &\leq C \frac{\|u(t)\|_{L^2}^2}{R^2} + 8 \iint_{\mathbb{R}^d \times \mathbb{R}^d} \left[\left(1 - \frac{1}{2} \frac{R}{|x|} \phi' \left(\frac{|x|}{R}\right)\right) x - \left(1 - \frac{1}{2} \frac{R}{|y|} \phi' \left(\frac{|y|}{R}\right)\right) y \right] \\ &\quad \times \frac{(x - y)}{|x - y|^6} |u(t, x)|^2 |u(t, y)|^2 dx dy. \end{aligned}$$

Let χ_Ω be the characteristic function of the set Ω . Note that

$$\left| \left(1 - \frac{1}{2} \frac{R}{|x|} \phi' \left(\frac{|x|}{R}\right)\right) x - \left(1 - \frac{1}{2} \frac{R}{|y|} \phi' \left(\frac{|y|}{R}\right)\right) y \right| \lesssim \chi_{\{|x| \geq R\} \cup \{|y| \geq R\}} |x - y|,$$

and the radial Sobolev inequality in [65, 67]

$$\|\chi_{\{|x| \geq R\}} u(t, \cdot)\|_{L^{\frac{2d}{d-2}}}^4 \lesssim \frac{1}{R^{\frac{4d-4}{d}}} \|u(t)\|_{\dot{H}^1}^{\frac{4}{d}} \|u(t)\|_{L^2}^{\frac{4d-4}{d}},$$

we have

$$\begin{aligned} &\iint_{\mathbb{R}^d \times \mathbb{R}^d} \chi_{\{|x| \geq R\} \cup \{|y| \geq R\}} \frac{1}{|x - y|^4} |u(t, x)|^2 |u(t, y)|^2 dx dy \\ &\lesssim \iint_{|x| \approx |y| \gtrsim R} \frac{1}{|x - y|^4} |u(t, x)|^2 |u(t, y)|^2 dx dy \\ &\quad + \iint_{\substack{\max(|x|, |y|) \gg \min(|x|, |y|) \\ \max(|x|, |y|) \geq R}} \frac{1}{|x - y|^4} |u(t, x)|^2 |u(t, y)|^2 dx dy \\ &\lesssim \|\chi_{\{|x| \geq R\}} u(t, \cdot)\|_{L^{\frac{2d}{d-2}}}^4 + \frac{1}{R^2} \|u(t)\|_{L^2}^2 \|u(t)\|_{\dot{H}^1}^2 \\ &\lesssim \frac{1}{R^{\frac{4d-4}{d}}} \|u(t)\|_{L^2}^{\frac{4d-4}{d}} \|u(t)\|_{\dot{H}^1}^{\frac{4}{d}} + \frac{1}{R^2} \|u(t)\|_{L^2}^2 \|u(t)\|_{\dot{H}^1}^2. \end{aligned}$$

This yields (5.6) by the mass conservation.

Step 2: Refined bound on $A_R(u(t))$ when $\delta(t)$ small. We claim that there exist δ_1 , $C > 0$ such that for any $R > 1$, and $t \geq 0$ with $\delta(t) \leq \delta_1$,

$$|A_R(u(t))| \leq C \left(\frac{1}{R^{\frac{d-2}{2}}} \delta(t) + \delta(t)^2 \right). \quad (5.7)$$

To do so, we first show that for small δ_1 ,

$$\mu_- := \inf \left\{ \mu(t), t \geq 0, \delta(t) \leq \delta_1 \right\} > 0. \quad (5.8)$$

Indeed, by (4.5) and Lemma 4.2, we have

$$u_{\theta(t), \mu(t)}(t) = W + V, \text{ with } \|V(t)\|_{\dot{H}^1} \approx \delta(t).$$

By the mass conservation, we have

$$\begin{aligned} \|u_0\|_{L^2}^2 &\geq \int_{|x| \leq \mu(t)} |u(t, x)|^2 dx = \frac{1}{\mu(t)^2} \int_{|x| \leq 1} |u_{\theta(t), \mu(t)}|^2 dx \\ &\gtrsim \frac{1}{\mu(t)^2} \left(\int_{|x| \leq 1} |W|^2 dx - \int_{|x| \leq 1} |V(t)|^2 dx \right). \end{aligned}$$

This together with

$$\|V(t)\|_{L^2(|x| \leq 1)} \lesssim \|V(t)\|_{L^{\frac{2d}{d-2}}(|x| \leq 1)} \lesssim \|V(t)\|_{\dot{H}^1} \lesssim \delta(t)$$

implies that

$$\|u_0\|_{L^2}^2 \geq \frac{1}{\mu(t)^2} \left(\int_{|x| \leq 1} |W|^2 dx - C\delta(t)^2 \right).$$

Choosing sufficiently small δ_1 , one easily gets (5.8). By (4.5) and the change of variable, we have

$$|A_R(u(t))| = |A_R((W + V)_{-\theta(t), 1/\mu(t)})| = |A_{R\mu(t)}(W + V)|.$$

Note that

$$A_{R\mu(t)}(W) = 0, \text{ and } \|\nabla W\|_{L^2(|x| \geq \rho)} \approx \|W\|_{L^{\frac{2d}{d-2}}(|x| \geq \rho)} \approx \rho^{-\frac{d-2}{2}} \text{ for } \rho \geq 1,$$

we have

$$\begin{aligned} |A_R(u(t))| &= |A_{R\mu(t)}(W + V) - A_{R\mu(t)}(W)| \\ &\lesssim \int_{|x| \geq R\mu(t)} |\nabla W \cdot \nabla V(t)| + |\nabla V(t)|^2 dx \\ &\quad + \int_{R\mu(t) \leq |x| \leq 2R\mu(t)} \frac{1}{(R\mu(t))^2} (|W \cdot V(t)| + |V(t)|^2) dx \\ &\quad + \iint_{\substack{\{|x| \geq R\mu(t)\} \\ \cup \{|y| \geq R\mu(t)\}}} \frac{1}{|x - y|^4} (|W(x)V(t, x)||W(y)|^2 + |W(x)|^2|W(y)V(t, y)| \\ &\quad + |W(x)|^2|V(t, y)|^2 + |V(t, x)|^2|W(y)|^2 \\ &\quad + |W(x)V(t, x)||V(t, y)|^2 + |V(t, x)|^2|V(t, y)|^2) dx dy \\ &\lesssim \frac{\|V(t)\|_{\dot{H}^1}^{\frac{d-2}{2}}}{(R\mu(t))^{\frac{d-2}{2}}} + \frac{\|V(t)\|_{\dot{H}^1}}{(R\mu(t))^{d-2}} + \|V(t)\|_{\dot{H}^1}^2 + \|V(t)\|_{\dot{H}^1}^3 + \|V(t)\|_{\dot{H}^1}^4. \end{aligned}$$

This combining with (5.8) implies that (5.7) if δ_1 is small enough.

Step 3: Conclusion. From (5.7), there exist $\delta_2 > 0$ and $R_2 \geq 1$ such that if $R \geq R_2$, $\delta(t) \leq \delta_2$,

$$\left| A_R(u(t)) \right| \leq C \left(R^{-\frac{d-2}{2}} \delta(t) + \delta(t)^2 \right) \leq 4\delta(t).$$

Let

$$f_{R_3}(\delta) := \frac{C}{R_3^2} + \frac{C}{R_3^{\frac{4d-4}{d}}} (\delta + \|W\|_{\dot{H}^1}^2)^{\frac{2}{d}} + \frac{C}{R_3^2} (\delta + \|W\|_{\dot{H}^1}^2) - 4\delta,$$

where C is given by (5.6). For sufficient large R_3 , $f_{R_3}(\delta)$ is concave on δ . Choosing R_3 large enough such that $f_{R_3}(\delta_3) \leq 0$, and $f'_{R_3}(\delta_3) \leq 0$, we have for any $R \geq R_3$, $\delta \geq \delta_3$

$$f_R(\delta) \leq f_{R_3}(\delta) \leq 0.$$

This implies that $A_R(u(t)) \leq 4\delta(t)$, so we conclude the proof of (5.4) with $R_1 = \max(R_2, R_3)$.

Finally, note that $V_R(t) > 0$, $\partial_t^2 V_R(t) < 0$ for $t > 0$ and u is defined on $[0, +\infty)$, we easily see that $\partial_t V_R(t) > 0$ by the convexity analysis. \square

Lemma 5.4. *Under the assumptions of Proposition 5.1, there exist $c > 0$, $C > 0$ such that for $R \geq R_1$ (which is given in Lemma 5.3), and all $t \geq 0$,*

$$\int_t^{+\infty} \delta(s) ds \leq C e^{-ct}. \quad (5.9)$$

Proof. Fix $R \geq R_1$. By (5.3), (5.4) and (5.5), we have

$$4 \int_t^T \delta(s) ds \leq - \int_t^T \partial_s^2 V_R(s) ds = \partial_t V_R(t) - \partial_t V_R(T) \leq C R^2 \delta(t).$$

Let $T \rightarrow +\infty$, we have

$$\int_t^{+\infty} \delta(s) ds \leq C \delta(t).$$

By the Gronwall inequality, we have (5.9). \square

5.2. Convergence of the modulation parameters. Let us show that

$$\lim_{t \rightarrow +\infty} \delta(t) = 0. \quad (5.10)$$

If (5.10) does not hold, by (5.9), there exist two increasing sequences $(t_n)_n$, $(t'_n)_n$ such that $t_n < t'_n$, $\delta(t_n) \rightarrow 0$, $\delta(t'_n) = \varepsilon_1$ for some $0 < \varepsilon_1 < \delta_0$, and

$$\forall t \in (t_n, t'_n), \quad 0 < \delta(t) < \varepsilon_1.$$

On $[t_n, t'_n]$, $\theta(t)$ and $\mu(t)$ are well-defined by Lemma 4.1. By Proposition 2.7, there exist θ_0 and $\mu_0 > 0$ such that

$$u(t_n) \longrightarrow W_{-\theta_0, 1/\mu_0} \quad \text{in } \dot{H}^1. \quad (5.11)$$

By the mass conservation, there exists $u_\infty \in L^2$ such that $u(t) \rightharpoonup u_\infty$ in L^2 , as $t \rightarrow +\infty$. Hence, we have

$$u(t) \rightharpoonup W_{-\theta_0, 1/\mu_0} \quad \text{in } L^2, \quad \text{as } t \rightarrow +\infty,$$

which implies that

$$\mu(t) \text{ is bounded on } \bigcup_n [t_n, t'_n]. \quad (5.12)$$

By Lemma 4.2 and (5.9), we have

$$\begin{aligned} |\alpha(t_n)| &\approx \delta(t_n) \rightarrow 0, \text{ as } n \rightarrow +\infty, \\ |\alpha(t'_n) - \alpha(t_n)| &= \left| \int_{t_n}^{t'_n} \alpha'(s) ds \right| \leq C \left| \int_{t_n}^{t'_n} \delta(s) ds \right| \leq C e^{-ct_n}. \end{aligned}$$

Therefore $\alpha(t'_n) \rightarrow 0$ as $n \rightarrow +\infty$, which contradicts the definition of t'_n .

Similarly by (5.10), we have that $\mu(t)$ is bounded for large t , and the parameter $\alpha(t)$ is well defined for large t , and

$$\begin{aligned} \delta(t) &\approx |\alpha(t)| = |\alpha(t) - \alpha(+\infty)| \\ &\leq C \int_t^{+\infty} |\alpha'(s)| ds \leq C \int_t^{+\infty} \mu^2(s) \delta(s) ds \leq C e^{-ct}, \end{aligned} \quad (5.13)$$

which, together with Lemma 4.2, implies that $\frac{1}{\mu(t)^2}$ satisfies the Cauchy criterion as $t \rightarrow +\infty$. Then we have

$$\lim_{t \rightarrow +\infty} \mu(t) = \mu_{+\infty} \in (0, +\infty],$$

This combining with the boundness of $\mu(t)$ precludes the case $\lim_{t \rightarrow +\infty} \mu(t) = +\infty$. Hence we have

$$\lim_{t \rightarrow +\infty} \mu(t) = \mu_{\infty} \in (0, +\infty). \quad (5.14)$$

Thus by Lemma 4.2, we have

$$\|u - W_{\theta(t), \mu(t)}\|_{\dot{H}^1} + |\alpha'(t)| + |\theta'(t)| \leq C \delta(t) \leq C e^{-ct}.$$

This yields (5.2). \square

5.3. Blowup for the negative times. It is a consequence of the positivity of $\partial_t V_R(t)$ in (5.3) and the time reversal symmetry. Suppose that u is also global for the negative time. Applying Lemma 5.2, Lemma 5.3 and Lemma 5.4 to $\bar{u}(-t)$, we know that they also hold for the negative times. Hence by (5.10), we know that

$$\lim_{t \rightarrow \pm\infty} \delta(t) = 0.$$

By Lemma 5.2 and Lemma 5.3, we know that $\partial_t V_R(t) > 0$ and $\partial_t V_R(t) \rightarrow 0$, as $t \rightarrow \pm\infty$. By Lemma 5.3, we have $\partial_t^2 V_R(t) < 0$. This implies that $\partial_t V_R(t) \equiv 0$. It is a contradiction, so we conclude the proof. \square

6. CONVERGENCE TO W FOR THE SUBCRITICAL THRESHOLD SOLUTION

In this section, we consider the radial subcritical threshold solution u of (1.1). Similar to that in Section 5, the following proposition will give the dynamics of W^- of Theorem 1.2 in the negative time and is also the first step in the proof of case (a) of Theorem 1.3.

Proposition 6.1. *Let $u \in \dot{H}^1(\mathbb{R}^d)$ be a radial solution of (1.1), and $I = (T_-, T_+)$ denote its maximal interval of existence. Assume that*

$$E(u_0) = E(W), \quad \|\nabla u_0\|_{L^2} < \|\nabla W\|_{L^2}, \quad (6.1)$$

Then

$$I = \mathbb{R}.$$

Furthermore, if u does not scatter for the positive times, that is,

$$\|u\|_{Z(0, +\infty)} = \infty, \quad (6.2)$$

then there exist $\theta_0 \in \mathbb{R}, \mu_0 > 0, c, C > 0$ such that

$$\|u - W_{\theta_0, \mu_0}\|_{\dot{H}^1} \leq Ce^{-ct}, \quad \forall t \geq 0, \quad (6.3)$$

and

$$\|u\|_{Z(-\infty, 0)} < \infty. \quad (6.4)$$

An analogous assertion holds on $(-\infty, 0]$.

6.1. Compactness properties. It is well known that the solution with

$$E(u) < E(W), \quad \|\nabla u_0\|_{L^2} < \|\nabla W\|_{L^2},$$

is global well-posed and scatters in both time directions [41, 51]. By Lemma 2.8, the concentration compactness principle and the stability theory as the proof of Proposition 4.2 in [51], we can show:

Lemma 6.2. *Let u be a radial solution of (1.1) satisfying*

$$E(u_0) < E(W), \quad \|\nabla u_0\|_{L^2} < \|\nabla W\|_{L^2}, \quad \|u\|_{Z(0, T_+)} = +\infty.$$

Then there exists a continuous functions $\lambda(t)$ such that the set

$$K := \{(u(t))_{\lambda(t)}, t \in [0, T_+)\} \quad (6.5)$$

is pre-compact in $\dot{H}^1(\mathbb{R}^d)$.

Let u be a solution of (1.1), and $\lambda(t)$ be as in Lemma 6.2. Consider δ_0 as in Section 4. The parameters $\theta(t)$, $\mu(t)$ and $\alpha(t)$ are defined for $t \in D_{\delta_0} = \{t : \delta(t) < \delta_0\}$. By (4.5) and Lemma 4.2, there exists a constant $C_0 > 0$ such that

$$\begin{aligned} \int_{\mu(t) \leq |x| \leq 2\mu(t)} |\nabla u(t, x)|^2 dx &= \int_{\mu(t) \leq |x| \leq 2\mu(t)} \frac{1}{\mu(t)^d} \left| e^{i\theta(t)} \nabla u \left(t, \frac{x}{\mu(t)} \right) \right|^2 dx \\ &\geq \int_{1 \leq |x| \leq 2} |\nabla W|^2 - C_0 \delta(t), \quad \forall t \in D_{\delta_0}. \end{aligned}$$

Taking a smaller δ_0 if necessary, we can assume that the right hand side of the above inequality is bounded from below by a positive constant ε_0 on D_{δ_0} . Thus, we have

$$\int_{\frac{\mu(t)}{\lambda(t)} \leq |x| \leq \frac{2\mu(t)}{\lambda(t)}} \frac{1}{\lambda(t)^d} \left| \nabla u \left(t, \frac{x}{\lambda(t)} \right) \right|^2 dx \geq \int_{1 \leq |x| \leq 2} |\nabla W|^2 - C_0 \delta(t), \quad \forall t \in D_{\delta_0}.$$

By the compactness of \overline{K} , it follows that for any $t \in D_{\delta_0}$, we have $|\mu(t)| \sim |\lambda(t)|$. As a consequence, we may modify $\lambda(t)$ such that K defined by (6.5) remains pre-compact in \dot{H}^1 and

$$\forall t \in D_{\delta_0}, \quad \lambda(t) = \mu(t). \quad (6.6)$$

As a consequence of Lemma 6.2, we have

Corollary 6.3. *Let u be a radial solution of (1.1) satisfying (6.1) and not scatter for the positive times. Then*

- (a) $T_+ = +\infty$.
- (b) $\lim_{t \rightarrow \infty} \sqrt{t} \lambda(t) = \infty$;

As a direct consequence of (a) in Corollary 6.3, we have

Corollary 6.4. *Let u be a radial solution of (1.1) with the maximal existence interval I such that*

$$E(u_0) \leq E(W), \quad \|\nabla u_0\|_2 \leq \|\nabla W\|_2,$$

then $I = \mathbb{R}$.

Proof. If $\|\nabla u_0\|_2 = \|\nabla W\|_2$, then by (2.5), we have $E(u_0) = E(W)$, then by the variational characterization of W , we have $u_0 = \pm W_{\theta_0, \lambda_0}$ for some $\theta_0 \in [0, 2\pi)$, $\lambda_0 > 0$. By uniqueness of (1.1), u is only the stationary solution $\pm W_{\theta_0, \lambda_0}$, which is globally defined.

If $\|\nabla u_0\|_2 < \|\nabla W\|_2$, $E(u_0) < E(W)$, then from [51], we have the solution u is global wellposed and scatters.

Now we consider the case $\|\nabla u_0\|_2 < \|\nabla W\|_2$, $E(u_0) = E(W)$. If $\|u\|_{Z(I)} < +\infty$, then by the finite blowup criterion, we know that u is a global solution. If $\|u\|_{Z([0, T_+))} = +\infty$, then by Corollary 6.3 (a), we have $T_+ = +\infty$. The same result holds for the negative time. \square

Proof of Corollary 6.3. We show (a) by contradiction. Assume that

$$T_+(u_0) < +\infty.$$

For this case we can show that

$$\lambda(t) \rightarrow +\infty, \quad \text{as } t \rightarrow T_+(u_0).$$

Then using the localized mass argument, the almost finite propagation speed and the compactness property of \overline{K} in \dot{H}^1 as Step 2 of Lemma 2.8 in [16], we can show that

$$u_0 \in L^2.$$

Moreover, we have $u \equiv 0$, which contradicts the assumption that $E(u_0) = E(W)$ or that u blows up at finite time $T_+ > 0$.

We also show (b) by the compactness argument. Assume that (b) does not hold. Then there exists a sequence $t_n \rightarrow +\infty$ such that

$$\lim_{t_n \rightarrow +\infty} \sqrt{t_n} \lambda(t_n) = \tau_0 \in [0, +\infty).$$

Consider

$$w_n(s, y) = \lambda(t_n)^{-\frac{d-2}{2}} u \left(t_n + \frac{s}{\lambda(t_n)^2}, \frac{y}{\lambda(t_n)} \right).$$

By the compactness of \overline{K} , up to a subsequence, there exists a function $w_0 \in \dot{H}^1$ such that

$$w_n(0) \rightarrow w_0 \text{ in } \dot{H}^1$$

Let w be the solution of (1.1) with initial data w_0 . Note that

$$E(u_0) = E(W), \quad \|\nabla u(t_n)\|_2 < \|\nabla W\|_2,$$

we have

$$E(w_0) = E(W), \quad \|\nabla w_0\|_2 \leq \|\nabla W\|_2.$$

Thus by Corollary 6.4, we have that $T_-(w_0) = T_+(w_0) = \infty$. By Theorem 2.4 and $-\sqrt{t_n} \lambda(t_n) \rightarrow -\tau_0$, we have

$$\lambda(t_n)^{-\frac{d-2}{2}} u_0 \left(\frac{y}{\lambda(t_n)} \right) = w_n(-t_n \lambda(t_n)^2, y) \rightarrow w(-\tau_0^2, y), \text{ in } \dot{H}^1.$$

Since $\lambda(t_n) \rightarrow 0$, we have

$$\lambda(t_n)^{-\frac{d-2}{2}} u_0 \left(\frac{y}{\lambda(t_n)} \right) \rightarrow 0, \text{ in } \dot{H}^1.$$

Thus $w(-\tau_0^2) = 0$, which contradicts $E(w) = E(W) > 0$. \square

6.2. Convergence in the ergodic mean.

Lemma 6.5. *Let u be a radial solution of (1.1) satisfying (6.1) and (6.2). Then*

$$\lim_{T \rightarrow +\infty} \frac{1}{T} \int_0^T \delta(t) dt = 0, \quad (6.7)$$

where $\delta(t)$ is defined by (4.4).

Proof. By Lemma 2.9 and the Hardy inequality, we have

$$|\partial_t V_R(t)| \leq CR^2.$$

From $E(u) = E(W) = \|\nabla W\|_{L^2}^2/4$, we have

$$\partial_t^2 V_R(t) = -8\delta(t) + A_R(t),$$

where $A_R(u(t))$ is defined in Lemma 2.9. By the compactness of \overline{K} in \dot{H}^1 , for any $\epsilon > 0$, there exists $\rho_\epsilon > 0$ such that

$$\int_{|x| \geq \frac{\rho_\epsilon}{\lambda(t)}} |\nabla u(t, x)|^2 dx \leq \epsilon.$$

Note that

$$\left| \left(1 - \frac{1}{2} \frac{R}{|x|} \phi' \left(\frac{|x|}{R} \right) \right) x - \left(1 - \frac{1}{2} \frac{R}{|y|} \phi' \left(\frac{|y|}{R} \right) \right) y \right| \lesssim \chi_{\{|x| \geq R\} \cup \{|y| \geq R\}} |x - y|,$$

and

$$\begin{aligned} & \iint \chi_{\{|x| \geq R\} \cup \{|y| \geq R\}} \frac{1}{|x - y|^4} |u(t, x)|^2 |u(t, y)|^2 dx dy \\ & \lesssim \|u(t)\|_{L^{\frac{2d}{d-2}}(|x| \geq R)}^2 \|u(t)\|_{\dot{H}^1}^2 \lesssim \|u(t)\|_{L^{\frac{2d}{d-2}}(|x| \geq R)}^2, \end{aligned}$$

we have for $R \geq \rho_\epsilon/\lambda(t)$

$$\forall t \geq 0, \quad |A_R(t)| \leq \epsilon. \quad (6.8)$$

Fix ϵ , choose ϵ_0 and M_0 such that

$$2C\epsilon_0^2 = \epsilon, \quad M_0\epsilon_0 \geq \rho_\epsilon.$$

By Corollary 6.3 (b), there exists $t_0 \geq 0$ such that

$$\forall t \geq t_0, \quad \lambda(t) \geq \frac{M_0}{\sqrt{t}}.$$

For $T \geq t_0$, let $R := \epsilon_0\sqrt{T}$. For $t \in [t_0, T]$, we have

$$R \geq \epsilon_0\sqrt{T} \frac{M_0}{\sqrt{t}\lambda(t)} = \frac{\sqrt{T}}{\sqrt{t}} \frac{M_0\epsilon_0}{\lambda(t)} \geq \frac{\rho_\epsilon}{\lambda(t)},$$

this yields that

$$8 \int_t^T \delta(s) ds \leq \int_t^T \partial_s^2 V_R(s) ds + |A_R(s)|(T - t_0) \leq 2CR^2 + \epsilon T = 2\epsilon T.$$

Let $T \rightarrow +\infty$, we have

$$\lim_{T \rightarrow +\infty} \frac{1}{T} \int_0^T \delta(s) ds \leq \frac{\epsilon}{4}.$$

This completes the proof. \square

Corollary 6.6. *Let u be a radial solution of (1.1) satisfying (6.1) and (6.2). Then there exists a sequence t_n such that $t_n \rightarrow +\infty$ and*

$$\lim_{n \rightarrow +\infty} \delta(t_n) = 0. \quad (6.9)$$

6.3. Exponential convergence.

Lemma 6.7. *Let u be a radial solution of (1.1) satisfying (6.1), (6.2) and (6.6), and $\lambda(t)$ be as in Lemma 6.2. Then there exists a constant C such that if $0 \leq \sigma < \tau$,*

$$\int_{\sigma}^{\tau} \delta(t) dt \leq C \left(\sup_{\sigma \leq t \leq \tau} \frac{1}{\lambda(t)^2} \right) (\delta(\sigma) + \delta(\tau)).$$

Proof. For $R > 0$, Let us consider the function $V_R(t)$ defined as in (2.6).

By the same estimate as that in Lemma 5.2, there is a constant C_0 independent of $t \geq 0$ such that

$$|\partial_t V_R(t)| \leq C_0 R^2 \delta(t). \quad (6.10)$$

Now we show that if $\epsilon > 0$, there exists R_{ϵ} such that for any $R \geq R_{\epsilon}/\lambda(t)$, then

$$\partial_t^2 V_R(t) \geq (8 - \epsilon) \delta(t). \quad (6.11)$$

Indeed, by Lemma 2.9 and $E(u) = E(W) = \|\nabla W\|_{L^2}^2/4$, we have

$$\begin{aligned} \partial_t^2 V_R(t) &= 8 \int_{\mathbb{R}^d} |\nabla u(t, x)|^2 dx - 8 \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|u(t, x)|^2 |u(t, y)|^2}{|x - y|^4} dx dy + A_R(u(t)) \\ &= 8\delta(t) + A_R(u(t)), \end{aligned}$$

where $A_R(u(t))$ is defined in Lemma 2.9.

To prove (6.11), it suffices to show that if $\epsilon > 0$, there exists R_{ϵ} such that for any $R \geq R_{\epsilon}/\lambda(t)$

$$|A_R(t)| \lesssim \epsilon \delta(t).$$

For the case $\delta(t) \geq \delta_0$, as the estimate (6.8), we can use the compactness of \overline{K} in \dot{H}^1 to show that for any $t \geq 0$, $\epsilon > 0$, there exists $\rho_{\epsilon} > 0$, such that for any $R \geq \rho_{\epsilon}/\lambda(t)$,

$$|A_R(t)| \leq \epsilon \lesssim \epsilon \delta(t).$$

For the case $\delta(t) < \delta_0$, similar to the proof of Step 2 in Lemma 5.3, we can show that for any $t \geq 0$, $\rho > 1$, there exists $C > 0$ such that for any $R \geq \rho/\lambda(t)$,

$$|A_R(t)| \leq C \left(\rho^{-\frac{d-2}{2}} \delta(t) + \delta(t)^2 \right).$$

This implies (6.11) if we choose ρ large enough.

By (6.11), there exists R_2 such that for any $R \geq R_2/\lambda(t)$

$$\partial_t^2 V_R(t) \geq 4\delta(t).$$

Finally, if taking $R = R_2 \sup_{\sigma \leq t \leq \tau} \left(\frac{1}{\lambda(t)} \right)$, and integrating between σ and τ , we have

$$4 \int_{\sigma}^{\tau} \delta(t) dt \leq \int_{\sigma}^{\tau} \partial_t^2 V_R(t) dt = \partial_t V_R(\tau) - \partial_t V_R(\sigma) \leq C R^2 (\delta(\tau) + \delta(\sigma)).$$

This finishes the proof of Lemma 6.7. \square

Lemma 6.8. *Let u be a radial solution of (1.1) satisfying (6.1), (6.2) and (6.6), and $\lambda(t)$ be as in Lemma 6.2. Then there is a constant $C_0 > 0$ such that for any $\sigma, \tau > 0$ with $\sigma + \frac{1}{\lambda(\sigma)^2} \leq \tau$, we have*

$$\left| \frac{1}{\lambda(\tau)^2} - \frac{1}{\lambda(\sigma)^2} \right| \leq C_0 \int_{\sigma}^{\tau} \delta(t) dt. \quad (6.12)$$

Proof. We divide it into three steps:

Step 1: Local constancy of $\lambda(t)$. By the compactness of \overline{K} , one easily show that there exists $C_1 > 0$ such that

$$\forall \sigma, \tau \geq 0, |\tau - \sigma| \leq \frac{1}{\lambda(\sigma)^2} \implies \frac{\lambda(\tau)}{\lambda(\sigma)} + \frac{\lambda(\sigma)}{\lambda(\tau)} \leq C_1. \quad (6.13)$$

Indeed, for any two sequences $\tau_n, \sigma_n \geq 0$ such that

$$|\tau_n - \sigma_n| \leq \frac{1}{\lambda(\sigma_n)^2}.$$

Up to subsequence, we may assume that

$$\lim_{n \rightarrow +\infty} \lambda^2(\sigma_n)(\tau_n - \sigma_n) = s_0 \in [-1, 1].$$

Consider

$$v_n(s, y) = \left(\lambda(\sigma_n) \right)^{-\frac{d-2}{2}} u \left(\frac{s}{\lambda(\sigma_n)^2} + \sigma_n, \frac{y}{\lambda(\sigma_n)} \right).$$

By the compactness of \overline{K} , up to subsequence, there exists $v_0 \in \dot{H}^1$ such that

$$v_n(0) \longrightarrow v_0 \text{ in } \dot{H}^1, \text{ as } n \rightarrow +\infty.$$

Thus $E(v_0) = E(W)$ and $\|\nabla v_0\|_{L^2} < \|\nabla W\|_{L^2}$. Let v be the solution of (1.1) with v_0 . By Corollary 6.4 and Theorem 2.4, v is globally defined and

$$\left(\lambda(\sigma_n) \right)^{-\frac{d-2}{2}} u \left(\tau_n, \frac{y}{\lambda(\sigma_n)} \right) = v_n(\lambda(\sigma_n)^2(\tau_n - \sigma_n), y) \longrightarrow v(s_0, y) \text{ in } \dot{H}^1 \text{ as } n \rightarrow +\infty.$$

In addition, by the compactness of \overline{K} , we know that $\left(\lambda(\tau_n) \right)^{-\frac{d-2}{2}} u \left(\tau_n, \frac{y}{\lambda(\tau_n)} \right)$ converges in \dot{H}^1 . Thus $\lambda(\tau_n)/\lambda(\sigma_n) + \lambda(\sigma_n)/\lambda(\tau_n)$ is bounded.

Step 2: Control of the variations of $\delta(t)$. Let δ_0 be as in Lemma 4.6. Using the local constancy of $\lambda(t)$ and the compactness of \overline{K} , we will show that for any $\tau > 0$, if

$$\sup_{t \in \left[\tau, \tau + \frac{1}{\lambda(\tau)^2} \right]} \delta(t) > \delta_0,$$

then there exists $\delta_1 > 0$ such that

$$\inf_{t \in \left[\tau, \tau + \frac{1}{\lambda(\tau)^2} \right]} \delta(t) > \delta_1. \quad (6.14)$$

Indeed, if not, we may find sequences τ_n, t_n and t'_n such that

$$t_n, t'_n \in \left[\tau_n, \tau_n + \frac{1}{\lambda(\tau_n)^2} \right], \quad \delta(t_n) \rightarrow 0, \quad \delta(t'_n) > \delta_0.$$

Consider

$$v_n(s, y) = \lambda(t_n)^{-\frac{d-2}{2}} u \left(\frac{s}{\lambda(t_n)^2} + t_n, \frac{y}{\lambda(t_n)} \right).$$

From the compactness of \overline{K} and $\delta(t_n) \rightarrow 0$, we may assume that

$$v_n(0) \longrightarrow W_{\lambda_0} \text{ in } \dot{H}^1 \text{ as } n \rightarrow +\infty.$$

By Step 1, we know that $\lambda(t_n)/\lambda(\tau_n) \leq C$, thus $|\lambda(t_n)^2(t_n - t'_n)| < C$ for some constant $C > 0$. Up to a subsequence, we may assume that

$$\lim_{n \rightarrow +\infty} \lambda(t_n)^2(t_n - t'_n) = s_0 \in [-C, C].$$

By Theorem 2.4, we know that

$$v_n\left(\lambda(t_n)^2(t_n - t'_n), y\right) = \lambda(t_n)^{-\frac{d-2}{2}} u\left(t'_n, \frac{y}{\lambda(t_n)}\right) \longrightarrow W_{\lambda_0} \text{ in } \dot{H}^1 \text{ as } n \rightarrow +\infty.$$

This contradicts $\delta(t'_n) > \delta_0$.

Step 3: End of the proof. We first show that there exists $C > 0$ such that

$$0 \leq \sigma \leq \tilde{\sigma} \leq \tilde{\tau} \leq \tau = \sigma + \frac{1}{C_1^2 \lambda(\sigma)^2} \implies \left| \frac{1}{\lambda(\tilde{\tau})^2} - \frac{1}{\lambda(\tilde{\sigma})^2} \right| \leq C \int_{\sigma}^{\tau} \delta(t) dt. \quad (6.15)$$

where $C_1 \geq 1$ is the constant defined in Step 1. Indeed, if $\delta(t) \leq \delta_0$ on $[\sigma, \tau]$, then by Lemma 4.2 and (6.6), we have

$$\left| \frac{1}{\lambda(\tilde{\tau})^2} - \frac{1}{\lambda(\tilde{\sigma})^2} \right| = \left| \int_{\tilde{\sigma}}^{\tilde{\tau}} \frac{\lambda'(t)}{\lambda(t)^3} dt \right| \leq \int_{\tilde{\sigma}}^{\tilde{\tau}} \left| \frac{\mu'(t)}{\mu(t)^3} \right| dt \leq C \int_{\sigma}^{\tau} \delta(t) dt.$$

Otherwise if there exists $t \in [\sigma, \tau]$ such that $\delta(t) > \delta_0$, then by Step 2, we know that $\delta(t) \geq \delta_1$ for all $t \in [\sigma, \tau]$. Note that

$$|\tilde{\sigma} - \tilde{\tau}| \leq \frac{1}{C_1^2 \lambda(\sigma)^2} \leq \frac{1}{\lambda(\tilde{\sigma})^2}.$$

By Step 1, we obtain

$$\left| \frac{1}{\lambda(\tilde{\tau})^2} - \frac{1}{\lambda(\tilde{\sigma})^2} \right| \leq \frac{2C_1^2}{\lambda(\tilde{\sigma})^2} \leq \frac{2C_1^4}{\lambda(\sigma)^2} = 2C_1^5 |\tau - \sigma| \leq \frac{2C_1^5}{\delta_1} \int_{\sigma}^{\tau} \delta(t) dt.$$

Dividing $[\sigma, \tau]$ into small subintervals, we can complete the proof. \square

Lemma 6.9. *Let u be a radial solution of (1.1) satisfying (6.1), (6.2) and (6.6), and $\lambda(t)$ be as in Lemma 6.2. Then there exists C such that for all $t \geq 0$,*

$$\int_t^{+\infty} \delta(s) ds \leq C e^{-ct}. \quad (6.16)$$

Proof. We first show that $\frac{1}{\lambda(t)^2}$ is bounded. By Lemma 6.7 and Lemma 6.8, there exists a constant $C_0 > 0$ such that for all $0 \leq \sigma \leq s < t \leq \tau$ with $s + \frac{1}{\lambda(s)^2} < t$, we have

$$\left| \frac{1}{\lambda(s)^2} - \frac{1}{\lambda(t)^2} \right| \leq C_0 \sup_{\sigma \leq t \leq \tau} \left(\frac{1}{\lambda(t)^2} \right) (\delta(\sigma) + \delta(\tau)). \quad (6.17)$$

By Corollary 6.6, there exist $t_n \rightarrow +\infty$ and $n_0 \in \mathbb{N}$ such that for $n \geq n_0$,

$$\delta(t_n) \leq \frac{1}{4C_0}.$$

Take $\sigma = s = t_{n_0}$, $\tau = t_n$, then

$$\forall t \in (t_{n_0} + \frac{1}{\lambda(t_{n_0})^2}, t_n] \implies \left| \frac{1}{\lambda(t_{n_0})^2} - \frac{1}{\lambda(t)^2} \right| \leq \frac{1}{2} \sup_{t \geq t_{n_0}} \frac{1}{\lambda(t)^2}.$$

Note that $t_n \rightarrow +\infty$ as $n \rightarrow +\infty$, we have

$$\sup_{t \geq t_{n_0} + \frac{1}{\lambda(t_{n_0})^2}} \frac{1}{\lambda(t)^2} \leq \frac{1}{2} \sup_{t \geq t_{n_0}} \frac{1}{\lambda(t)^2} + \frac{1}{\lambda(t_{n_0})^2}.$$

Thus

$$\sup_{t \geq t_{n_0} + \frac{1}{\lambda(t_{n_0})^2}} \frac{1}{\lambda(t)^2} \leq \sup_{t_{n_0} \leq t \leq t_{n_0} + \frac{1}{\lambda(t_{n_0})^2}} \frac{1}{\lambda(t)^2} + \frac{2}{\lambda(t_{n_0})^2}.$$

This shows the boundness of $\frac{1}{\lambda(t)^2}$.

By Lemma 6.7 and the boundness of $\frac{1}{\lambda(t)^2}$, we have for $t + \frac{1}{\lambda(t)^2} < t_n$

$$\int_t^{t_n} \delta(s) ds \leq C(\delta(t) + \delta(t_n)).$$

Let $n \rightarrow +\infty$, we obtain

$$\int_t^\infty \delta(s) ds \leq C\delta(t).$$

This together with the Gronwall inequality yields (6.16). \square

6.4. Convergence of $\lambda(t)$. Now by Lemma 6.8 and (6.16), we have for $\sigma + \frac{1}{\lambda(\sigma)^2} < \tau$

$$\left| \frac{1}{\lambda(\sigma)^2} - \frac{1}{\lambda(\tau)^2} \right| \leq Ce^{-c\sigma}.$$

By means of the Cauchy criteria of convergence at infinity, there exists $\lambda_\infty \in (0, +\infty]$ such that

$$\left| \frac{1}{\lambda(t)^2} - \frac{1}{\lambda_\infty^2} \right| \leq Ce^{-ct}.$$

Now we show that

$$\lambda_\infty \in (0, +\infty). \quad (6.18)$$

Assume that $\lambda_\infty = +\infty$. Let $0 \leq \sigma \leq s$. By (6.16), there exists a sequence t_n such that

$$\delta(t_0) \leq \frac{1}{2C_0}, \quad \delta(t_n) \rightarrow 0 \quad \text{as } t_n \rightarrow +\infty.$$

For larger n , we have $s + \frac{1}{\lambda(s)^2} < t_n$. By Lemma 6.7, we obtain

$$\left| \frac{1}{\lambda(s)^2} - \frac{1}{\lambda(t_n)^2} \right| \leq C_0 \sup_{\sigma \leq t \leq t_n} \left(\frac{1}{\lambda(t)^2} \right) (\delta(\sigma) + \delta(t_n)).$$

Let $n \rightarrow +\infty$, we have

$$\sup_{t \geq \sigma} \frac{1}{\lambda(t)^2} \leq C_0 \delta(\sigma) \sup_{t \geq \sigma} \frac{1}{\lambda(t)^2}.$$

If taking $\sigma = t_0$, we have

$$\forall t \geq t_0, \quad \lambda(t) \equiv +\infty.$$

This contradicts the continuity of $\lambda(t)$ on \mathbb{R} .

6.5. Convergence of the modulation parameters. By (6.16), there exists $t_n \rightarrow +\infty$ such that

$$\delta(t_n) \rightarrow 0.$$

Fix such $\{t_n\}_{n \in \mathbb{N}}$. In the similar proof as leading to (5.10), one easily sees that

$$\lim_{t \rightarrow +\infty} \delta(t) = 0. \quad (6.19)$$

Now for large t , $\alpha(t)$ is well defined by Lemma 4.2, (6.6), (6.18), and (6.19). Furthermore, we also have

$$\begin{aligned} \delta(t) &\approx |\alpha(t)| = |\alpha(t) - \alpha(+\infty)| \\ &\leq C \int_t^{+\infty} |\alpha'(s)| ds \leq C \int_t^{+\infty} \mu^2(s) \delta(s) ds \leq C e^{-ct}. \end{aligned} \quad (6.20)$$

Finally, by Lemma 4.2, (6.6), (6.18), and the Cauchy criteria of convergence, there exists θ_∞ such that for large t

$$|\theta(t) - \theta_\infty| \leq C \int_t^\infty |\theta'(s)| ds \leq C \int_t^\infty \mu^2(s) \delta(s) ds \leq C e^{-ct}.$$

6.6. Scattering for the negative times. By Corollary 6.4, we know that u is globally well defined. Assume that u does not scatter for the negative time. Then by the analogue estimates as those in Subsection 6.1-6.3, we have

(a) there exists $\lambda(t)$, defined for $t \in \mathbb{R}$, such that the set

$$K := \left\{ (u(t))_{\lambda(t)}, t \in \mathbb{R} \right\}$$

is pre-compact in \dot{H}^1 .

(b) there exists an decreasing sequence $t'_n \rightarrow -\infty$ as $n \rightarrow +\infty$, such that

$$\lim_{n \rightarrow +\infty} \delta(t'_n) = 0.$$

(c) there is a constant $C > 0$ such that if $-\infty < \sigma < \tau < \infty$,

$$\int_\sigma^\tau \delta(t) dt \leq C \left(\sup_{\sigma \leq t \leq \tau} \frac{1}{\lambda(t)^2} \right) (\delta(\sigma) + \delta(\tau)).$$

(d) there is a constant $C > 0$ such that for any σ, τ with $\sigma + \frac{1}{\lambda(\sigma)^2} \leq \tau$, we have

$$\left| \frac{1}{\lambda(\tau)^2} - \frac{1}{\lambda(\sigma)^2} \right| \leq C \int_\sigma^\tau \delta(t) dt.$$

From (b)-(d), we know that

$$\lim_{t \rightarrow \pm\infty} \delta(t) = 0.$$

Note that $\frac{1}{\lambda(t)^2}$ is bounded for $t \in \mathbb{R}$, we easily verify that

$$\int_\sigma^\tau \delta(s) ds \leq C (\delta(\sigma) + \delta(\tau)).$$

Let $\sigma \rightarrow -\infty$ and $\tau \rightarrow +\infty$, we have

$$\delta(t) = 0, \forall t \in \mathbb{R}.$$

Thus $u = W$ up to the symmetry of (1.1), it contradicts $\|u_0\|_{\dot{H}^1} < \|W\|_{\dot{H}^1}$. \square

7. UNIQUENESS AND THE CLASSIFICATION RESULT

7.1. Exponentially small solutions of the linearized equation. Recall the notation of Section 3, in particular the operator \mathcal{L} and its eigenvalues and eigenfunctions.

Consider the radial functions $v(t)$ and $g(t)$

$$v \in C^0([t_0, +\infty), \dot{H}^1), \quad g \in C^0([t_0, +\infty), L^{\frac{2d}{d+2}}) \quad \text{and} \quad \nabla g \in N(t_0, +\infty)$$

satisfying

$$\partial_t v + \mathcal{L}v = g, \quad (x, t) \in \mathbb{R}^d \times [t_0, +\infty), \quad (7.1)$$

$$\|v(t)\|_{\dot{H}^1} \leq Ce^{-c_1 t}, \quad \|g(t)\|_{L^{\frac{2d}{d+2}}(\mathbb{R}^d)} + \|\nabla g(t)\|_{N(t, +\infty)} \leq Ce^{-c_2 t}, \quad (7.2)$$

where $0 < c_1 < c_2$. For any $c > 0$, we denote by c^- a positive number arbitrary close to c and such that $0 < c^- < c$.

By the Strichartz estimate and Lemma 2.11, we have

Lemma 7.1. *Under the above assumptions, then for any (q, r) with $\frac{2}{q} = d(\frac{1}{2} - \frac{1}{r})$, $q \in [2, +\infty]$, we have*

$$\|v\|_{l(t, +\infty)} + \|\nabla v\|_{L^q(t, +\infty; L^r)} \leq Ce^{-c_1 t}.$$

By the coercivity of the linearized energy in G_\perp , we will show that v must decay almost as fast as g , except in the direction \mathcal{Y}_+ where the decay is $e^{-e_0 t}$.

Proposition 7.2. *The following estimates hold.*

(a) *if $c_2 \leq e_0$ or $e_0 < c_1$, then*

$$\|v(t)\|_{\dot{H}^1} \leq Ce^{-c_2^- t}, \quad (7.3)$$

(b) *If $c_2 > e_0$, then there exists $a \in \mathbb{R}$ such that*

$$\|v(t) - ae^{-e_0 t} \mathcal{Y}_+\|_{\dot{H}^1} \leq Ce^{-c_2^- t}. \quad (7.4)$$

Proof. We decompose v as

$$v(t) = \alpha_+(t) \mathcal{Y}_+ + \alpha_-(t) \mathcal{Y}_- + \beta(t) iW + \gamma(t) \widetilde{W} + v_\perp(t), \quad v_\perp(t) \in G_\perp \cap \dot{H}_{rad}^1. \quad (7.5)$$

By (2.24), we can normalize the eigenfunction \mathcal{Y}_\pm such that

$$B(\mathcal{Y}_+, \mathcal{Y}_-) = 1.$$

By Remark 2.16, we have

$$\alpha_-(t) = B(v, \mathcal{Y}_+), \quad \alpha_+(t) = B(v, \mathcal{Y}_-), \quad (7.6)$$

$$\beta(t) = \frac{1}{\|W\|_{\dot{H}^1}^2} \left(v - \alpha_+(t) \mathcal{Y}_+ - \alpha_-(t) \mathcal{Y}_-, iW \right)_{\dot{H}^1}, \quad (7.7)$$

$$\gamma(t) = \frac{1}{\|W\|_{\dot{H}^1}^2} \left(v - \alpha_+(t) \mathcal{Y}_+ - \alpha_-(t) \mathcal{Y}_-, \widetilde{W} \right)_{\dot{H}^1}. \quad (7.8)$$

Step 1: Differential equations on the coefficients. We first show that

$$\frac{d}{dt} (e^{-e_0 t} \alpha_-) = e^{-e_0 t} B(g, \mathcal{Y}_+), \quad \frac{d}{dt} (e^{e_0 t} \alpha_+) = e^{e_0 t} B(g, \mathcal{Y}_-), \quad (7.9)$$

$$\frac{d}{dt} \beta(t) = \frac{(iW, \widetilde{v})_{\dot{H}^1}}{\|W\|_{\dot{H}^1}^2}, \quad \frac{d}{dt} \gamma(t) = \frac{(\widetilde{W}, \widetilde{v})_{\dot{H}^1}}{\|\widetilde{W}\|_{\dot{H}^1}^2}, \quad (7.10)$$

$$\frac{d}{dt}\Phi(v(t)) = 2B(g, v), \quad (7.11)$$

where

$$\tilde{v} = g - B(\mathcal{Y}_-, g)\mathcal{Y}_+ - B(\mathcal{Y}_+, g)\mathcal{Y}_- - \mathcal{L}v_\perp.$$

Indeed, by (7.1) and (7.6) and Remark 2.16, we have

$$\begin{aligned} \alpha'_-(t) &= B(\partial_t v, \mathcal{Y}_+) = B(-\mathcal{L}v, \mathcal{Y}_+) + B(g, \mathcal{Y}_+) \\ &= e_0 B(v, \mathcal{Y}_+) + B(g, \mathcal{Y}_+) = e_0 \alpha_-(t) + B(g, \mathcal{Y}_+), \end{aligned} \quad (7.12)$$

and

$$\begin{aligned} \alpha'_+(t) &= B(\partial_t v, \mathcal{Y}_-) = B(-\mathcal{L}v, \mathcal{Y}_-) + B(g, \mathcal{Y}_-) \\ &= -e_0 B(v, \mathcal{Y}_-) + B(g, \mathcal{Y}_-) = -e_0 \alpha_+(t) + B(g, \mathcal{Y}_-). \end{aligned} \quad (7.13)$$

This jointed with (7.12) implies (7.9).

By (7.1), (7.5), (7.7), (7.8), (7.12) and (7.13), we have

$$\begin{aligned} \frac{d}{dt}\beta(t) &= \frac{1}{\|W\|_{\dot{H}^1}^2} \left(\partial_t v - \alpha'_+(t)\mathcal{Y}_+ - \alpha'_-(t)\mathcal{Y}_-, iW \right)_{\dot{H}^1} \\ &= \frac{1}{\|W\|_{\dot{H}^1}^2} \left(g - \mathcal{L}v - \alpha'_+(t)\mathcal{Y}_+ - \alpha'_-(t)\mathcal{Y}_-, iW \right)_{\dot{H}^1} \\ &= \frac{1}{\|W\|_{\dot{H}^1}^2} \left(g - B(g, \mathcal{Y}_-)\mathcal{Y}_+ - B(g, \mathcal{Y}_+)\mathcal{Y}_- + \mathcal{L}v_\perp, iW \right)_{\dot{H}^1} := \frac{(\tilde{v}, iW)_{\dot{H}^1}}{\|W\|_{\dot{H}^1}^2}, \end{aligned}$$

and

$$\begin{aligned} \frac{d}{dt}\gamma(t) &= \frac{1}{\|\widetilde{W}\|_{\dot{H}^1}^2} \left(\partial_t v - \alpha'_+(t)\mathcal{Y}_+ - \alpha'_-(t)\mathcal{Y}_-, \widetilde{W} \right)_{\dot{H}^1} \\ &= \frac{1}{\|\widetilde{W}\|_{\dot{H}^1}^2} \left(g - B(g, \mathcal{Y}_-)\mathcal{Y}_+ - B(g, \mathcal{Y}_+)\mathcal{Y}_- + \mathcal{L}v_\perp, \widetilde{W} \right)_{\dot{H}^1} := \frac{(\tilde{v}, \widetilde{W})_{\dot{H}^1}}{\|\widetilde{W}\|_{\dot{H}^1}^2}, \end{aligned}$$

By (7.1), we have

$$\frac{d}{dt}\Phi(v) = \frac{d}{dt}B(v, v) = 2B(v, \partial_t v) = 2B(v, -\mathcal{L}v) + 2B(v, g) = 2B(v, g).$$

Step 2: Decay estimate on $\alpha_\pm(t)$. We now claim that there exists a real number $a \in \mathbb{R}$, such that

$$|\alpha_-(t)| \leq Ce^{-c_2 t}, \quad (7.14)$$

$$|\alpha_+(t)| \leq Ce^{-c_2^- t} \quad \text{if } e_0 \leq c_1 \text{ or } c_2 \leq e_0, \quad (7.15)$$

$$|\alpha_+(t) - ae^{-e_0 t}| \leq Ce^{-c_2 t} \quad \text{if } c_1 \leq e_0 < c_2. \quad (7.16)$$

By the definition of (2.19), we have

$$\begin{aligned} 2B(g, h) &= \int_{\mathbb{R}^d} (L_+ g_1) h_1 \, dx + \int_{\mathbb{R}^d} (L_- g_2) h_2 \, dx \\ &= \int_{\mathbb{R}^d} \nabla g_1 \nabla h_1 \, dx - \int_{\mathbb{R}^d} \left(\frac{1}{|x|^4} * |W|^2 \right) g_1 h_1 \, dx - 2 \int_{\mathbb{R}^d} \left(\frac{1}{|x|^4} * (W g_1) \right) W h_1 \, dx \\ &\quad + \int_{\mathbb{R}^d} \nabla g_2 \nabla h_2 \, dx - \int_{\mathbb{R}^d} \left(\frac{1}{|x|^4} * |W|^2 \right) g_2 h_2 \, dx. \end{aligned}$$

Hence, for any time-interval I with $|I| < \infty$, we have

$$\begin{aligned} \int_I \left| B(g(t), \mathcal{Y}_\pm) \right| dt &\lesssim |I|^{\frac{1}{3}} \|\nabla g\|_{N(I)} \|\nabla \mathcal{Y}_\pm\|_{L_x^{\frac{6d}{3d-4}}(\mathbb{R}^d)} \\ &\quad + |I| \cdot \|g\|_{L_I^\infty L_x^{\frac{2d}{d+2}}(\mathbb{R}^d)} \|\mathcal{Y}_\pm\|_{L_I^\infty L_x^{\frac{2d}{d-2}}(\mathbb{R}^d)} \|W\|_{L_x^{\frac{2d}{d-4}}(\mathbb{R}^d)}^2. \end{aligned}$$

This together with (7.2) implies that

$$\int_t^{t+1} \left| e^{-e_0 s} B(g(s), \mathcal{Y}_+) \right| ds \leq C e^{-e_0 t} e^{-c_2 t}.$$

By Lemma 2.11, we have

$$\int_t^{+\infty} \left| e^{-e_0 s} B(g(s), \mathcal{Y}_+) \right| ds \leq C e^{-(e_0+c_2)t}. \quad (7.17)$$

By (7.2), we know that $e^{-e_0 t} \alpha_-(t)$ tends to 0 when t goes to infinity. Integrating the equation on α_- in (7.9) between t and $+\infty$, we have

$$|\alpha_-(t)| \leq C e^{-c_2 t}.$$

Now we prove (7.15). First consider the case $e_0 < c_1$. By (7.2), we know that $e^{e_0 t} \alpha_+(t)$ tends to 0 when t goes to infinity. By (7.2) and the similar estimate as (7.17), we also have

$$\int_t^{+\infty} \left| e^{e_0 s} B(g(s), \mathcal{Y}_-) \right| ds \leq C e^{(e_0-c_2)t}.$$

Integrating the equation on α_+ in (7.9) between t and $+\infty$, we have

$$|\alpha_+(t)| \leq C e^{-c_2 t}.$$

If $c_1 \leq e_0 < c_2$, by (7.2), we have

$$\int_t^{t+1} \left| e^{e_0 s} B(g(s), \mathcal{Y}_-) \right| ds \leq C e^{e_0 t} e^{-c_2 t},$$

which together with Lemma 2.11 implies that

$$\int_{t_0}^{+\infty} \left| e^{e_0 s} B(g(s), \mathcal{Y}_-) \right| ds \lesssim e^{e_0 t_0} e^{-c_2 t_0} < \infty.$$

By (7.9), we know that $e^{e_0 t} \alpha_+(t)$ satisfies the Cauchy criterion as $t \rightarrow +\infty$, then

$$\lim_{t \rightarrow +\infty} e^{e_0 t} \alpha_+(t) = a,$$

and

$$|e^{e_0 t} \alpha_+(t) - a| \leq C e^{e_0 t} e^{-c_2 t}.$$

This shows (7.16).

If $c_1 < c_2 \leq e_0$, integrating the equation on α_+ in (7.9) between 0 and t , we have

$$\alpha_+(t) = e^{-e_0 t} \alpha_+(0) + e^{-e_0 t} \int_0^t e^{e_0 s} B(g(s), \mathcal{Y}_-) ds,$$

by (7.2), we know that

$$\left| \int_0^t e^{e_0 s} B(g(s), \mathcal{Y}_-) ds \right| \leq \begin{cases} C e^{(e_0-c_2)t}, & \text{if } c_2 < e_0, \\ Ct, & \text{if } c_2 = e_0. \end{cases}$$

This yields (7.15) in this case.

Step 3: Proof of the case $c_2 \leq e_0$, or $c_2 > e_0, a = 0$. By Step 2, we know in this case that

$$|\alpha_+(t)| + |\alpha_-(t)| \lesssim e^{-c_2^- t}. \quad (7.18)$$

Since

$$\int_t^{t+1} B(g(s), v(s)) ds \leq C e^{-(c_1+c_2)t},$$

we have by Lemma 2.11

$$\int_t^{+\infty} B(g, v) ds \leq C e^{-(c_1+c_2)t}.$$

By (7.11) and $\|\Phi(v(t))\| \lesssim \|v(t)\|_{\dot{H}^1}^2 \rightarrow 0$ as $t \rightarrow +\infty$, we have

$$|\Phi(v(t))| \leq C e^{-(c_1+c_2)t}.$$

Note that

$$\Phi(v) = B(v, v) = B(v_\perp, v_\perp) + 2\alpha_+\alpha_-,$$

we obtain from Lemma 2.15 and (7.18)

$$\|v_\perp(t)\|_{\dot{H}^1}^2 \lesssim |B(v_\perp, v_\perp)| \leq C e^{-(c_1+c_2)t}. \quad (7.19)$$

Now we turn to estimate the decay of $\beta(t)$. First by (7.7) and Step 2, we know that

$$|\beta(t)| \rightarrow 0, \text{ as } t \rightarrow +\infty.$$

In addition, by $\mathcal{L}^*(i\Delta W) = L_+(\Delta W) \in L^{\frac{2d}{d+2}}(\mathbb{R}^d)$, we also have

$$\begin{aligned} \int_t^{t+1} |(\tilde{v}, iW)_{\dot{H}^1}| ds &\lesssim e^{-c_2 t} + \int_t^{t+1} |(iW, \mathcal{L}v_\perp(s))_{\dot{H}^1}| ds \\ &\lesssim e^{-c_2 t} + \int_t^{t+1} \int_{\mathbb{R}^d} |\mathcal{L}^*(i\Delta W) \overline{v_\perp}(s)| dx ds \\ &\lesssim e^{-c_2 t} + \|v_\perp(t)\|_{L^\infty \dot{H}^1} \lesssim e^{-\frac{c_1+c_2}{2}t}. \end{aligned}$$

This, together with (7.10) and Lemma 2.11, implies that

$$|\beta(t)| \lesssim e^{-\frac{c_1+c_2}{2}t}. \quad (7.20)$$

By the analogue analysis, we can obtain the estimate of $\gamma(t)$

$$|\gamma(t)| \lesssim e^{-\frac{c_1+c_2}{2}t}. \quad (7.21)$$

Thus v, g satisfies (7.1) and (7.2) with c_1 replaced by $c_1' = \frac{c_1+c_2}{2}$. An iteration argument gives

$$\|v(t)\|_{\dot{H}^1} \leq C e^{-c_2^- t}.$$

Hence we conclude the proof of the case $c_2 \leq e_0$ or $c_2 > e_0, a = 0$.

Step 4: Proof of the case $c_2 > e_0$, and $a \neq 0$. By Step 2, if $c_1 > e_0$, we can take $a = 0$, so that we may assume that $c_1 \leq e_0$. Let

$$\bar{v}(t) := v(t) - a e^{-e_0 t} \mathcal{Y}_+,$$

then

$$\partial_t \bar{v}(t) + \mathcal{L} \bar{v}(t) = g(t), \quad \|\bar{v}(t)\|_{\dot{H}^1} \leq C e^{-c_1 t}$$

and by (7.16), we have

$$\lim_{t \rightarrow +\infty} e^{e_0 t} \bar{\alpha}_+(t) = 0,$$

where $\bar{\alpha}_+(t) = B(\bar{v}(t), \mathcal{Y}_-)$ is the coefficient of \mathcal{Y}_+ in the decomposition of $\bar{\alpha}_+(t)$. Thus $\bar{\alpha}_+(t)$ and g satisfy all the assumptions of Step 3, hence, we have

$$\|v(t) - ae^{-e_0 t} \mathcal{Y}_+\|_{\dot{H}^1} \leq Ce^{-c_2^- t}.$$

This completes the proof. \square

7.2. Uniqueness.

Proposition 7.3. *Let u be a radial solution of (1.1), defined on $[t_0, +\infty)$, such that $E(u) = E(W)$ and*

$$\exists c, C > 0, \|u - W\|_{\dot{H}^1} \leq Ce^{-ct}, \forall t \geq t_0. \quad (7.22)$$

Then there exists a unique $a \in \mathbb{R}$ such that $u = U^a$ is the solution of (1.1) defined in Proposition 3.3.

Proof. Let $u = W + v$ be a radial solution of (1.1) for $t \geq t_0$ satisfying (7.22). Then v satisfies the linearized equation (2.9), that is,

$$\partial_t v + \mathcal{L}v = R(v), \quad (7.23)$$

Step 1: We will show that there exists $A \in \mathbb{R}$ such that

$$\forall \eta > 0, \|v(t) - ae^{-e_0 t} \mathcal{Y}_+\|_{\dot{H}^1} + \|v(t) - ae^{-e_0 t} \mathcal{Y}_+\|_{l(t, +\infty)} \leq C_\eta e^{-2^- e_0 t}. \quad (7.24)$$

Indeed, we only need show

$$\|v(t)\|_{\dot{H}^1} \leq Ce^{-e_0 t}, \quad \|R(v(t))\|_{L^{\frac{2d}{d+2}}(\mathbb{R}^d)} + \|\nabla R(v(t))\|_{N(t, +\infty)} \leq Ce^{-2e_0 t}, \quad (7.25)$$

which together with Proposition 7.2 gives (7.24). By Lemma 2.10 and Lemma 2.12 it suffices to show the first estimate.

By (7.22), Lemma 2.10 and Lemma 2.12, we have

$$\|R(v(t))\|_{L^{\frac{2d}{d+2}}(\mathbb{R}^d)} + \|\nabla R(v(t))\|_{N(t, +\infty)} \leq Ce^{-2ct}.$$

Then Proposition 7.2 gives that

$$\|v(t)\|_{\dot{H}^1} \leq C \left(e^{-e_0 t} + e^{-\frac{3}{2} ct} \right).$$

If $\frac{3}{2} c \geq e_0$, we obtain the first inequality in (7.25). If not, an iteration argument gives the first inequality in (7.25).

Step 2: We will use the induction argument to show that

$$\forall m > 0, \|u(t) - U^a(t)\|_{\dot{H}^1} + \|u(t) - U^a(t)\|_{l(t, +\infty)} \leq e^{-mt}. \quad (7.26)$$

By Proposition 3.3 and Step 1, (7.26) holds with $m = \frac{3}{2} e_0$. Let us assume that (7.26) holds with $m = m_1 > e_0$, we will show that it also holds for $m = m_1 + \frac{1}{2} e_0$, which yields (7.26) by iteration.

Let $u^a(t) = U^a - W$, then we have

$$\partial_t(v - u^a) + \mathcal{L}(v - u^a) = -R(v) + R(u^a).$$

By assumption, we have

$$\|v(t) - u^a(t)\|_{\dot{H}^1} + \|v(t) - u^a(t)\|_{l(t, +\infty)} \leq Ce^{-m_1 t}.$$

By Lemma 2.10 and Lemma 2.11, we have

$$\|R(v(t)) - R(u^a(t))\|_{L^{\frac{2d}{d+2}}(\mathbb{R}^d)} + \|\nabla(R(v(t)) - R(u^a(t)))\|_{N(t,+\infty)} \leq Ce^{-(m_1+e_0)t}.$$

By Proposition 7.2 and Lemma 7.1, we have

$$\forall m > 0, \|v(t) - u^a(t)\|_{\dot{H}^1} + \|v(t) - u^a(t)\|_{l(t,+\infty)} \leq Ce^{-(m_1+\frac{3}{4}e_0)t} \leq e^{-(m_1+\frac{1}{2}e_0)t}.$$

Step 3: End of the proof. Using (7.26) with $m = (k_0 + 1)e_0$, (where k_0 is given by Proposition 3.3), we get that for large $t > 0$,

$$\|u(t) - U_{k_0}^a(t)\|_{l(t,+\infty)} \leq e^{-(k_0+\frac{1}{2})t}.$$

By the uniqueness in Proposition 3.3, we get that $u = U^a$, so we complete the proof of Proposition . \square

Corollary 7.4. *For any $a \neq 0$, there exists $T_a \in \mathbb{R}$ such that*

$$\begin{cases} U^a(t) = W^+(t - T_a), & \text{if } a > 0, \\ U^a(t) = W^-(t - T_a), & \text{if } a < 0. \end{cases} \quad (7.27)$$

Proof. Let $a \neq 0$ and choose T_a such that $|a|e^{-e_0 T_a} = 1$. By (3.7), we have

$$\begin{aligned} & \|U^a(t + T_a) - W - \text{sign}(a)e^{-e_0 t}\mathcal{Y}_+\|_{\dot{H}^1} \\ &= \|U^a(t + T_a) - W - ae^{-e_0(t+T_a)}\mathcal{Y}_+\|_{\dot{H}^1} \leq e^{-\frac{3}{2}e_0(t+T_a)} \leq Ce^{-\frac{3}{2}e_0 t}. \end{aligned} \quad (7.28)$$

In addition, $U^a(t + T_a)$ satisfies the assumption of Proposition 7.2, this implies that there exists \tilde{a} such that

$$U^a(t + T_a) = U^{\tilde{a}}(t).$$

By (7.28) and (3.7), we know that $\tilde{a} = 1$ if $a > 0$ and $\tilde{a} = -1$ if $a < 0$, this completes the proof. \square

7.3. Proof of Theorem 1.3. Let us first show (a). Let u be a radial solution to (1.1) on $[-T_-, T_+]$ such that, replacing if necessary $u(t)$ by $\bar{u}(-t)$

$$E(u_0) = E(W), \|\nabla u_0\|_{L^2} < \|\nabla W\|_{L^2}, \text{ and } \|u\|_{Z(0, T_+)} = +\infty.$$

Then by Proposition 6.1, we know that $T_- = T_+ = \infty$ and there exist $\theta_0 \in \mathbb{R}, \mu_0 > 0$ and $c, C > 0$ such that

$$\|u(t) - W_{\theta_0, \mu_0}\|_{\dot{H}^1} \leq Ce^{-ct}.$$

This shows that $u_{-\theta_0, \frac{1}{\mu_0}}$ fulfills the assumptions of Proposition 7.2. By $\|u\|_{\dot{H}^1} < \|W\|_{\dot{H}^1}$ and Corollary 7.4, we know that there exist $a < 0$ and T_a such that

$$u_{-\theta_0, \frac{1}{\mu_0}}(t) = U^a(t) = W^-(t - T_a),$$

which shows (a).

(b) is a direct consequence of the variational characterization of W .

The proof of (c) is similar to that of (a). Let u be a radial solution of (1.1) defined on $[0, +\infty)$ (Replacing if necessary $u(t)$ by $\bar{u}(-t)$) such that

$$E(u) = E(W), \|\nabla u_0\|_{L^2} > \|\nabla W\|_{L^2}, \text{ and } u_0 \in L^2.$$

Due to Proposition 5.1, there exist $\theta_0 \in \mathbb{R}, \mu_0 > 0$ and $c, C > 0$ such that

$$\|u(t) - W_{\theta_0, \mu_0}\|_{\dot{H}^1} \leq Ce^{-ct}.$$

Similar to the proof of (a), we know that there exist $a > 0$ and T_a such that

$$u_{-\theta_0, \frac{1}{\mu_0}}(t) = U^a(t) = W^+(t - T_a).$$

This shows (c). The proof of Theorem 1.3 is thus complete. \square

APPENDIX A. UNIQUENESS OF THE GROUND STATE IN $L^{\frac{2d}{d-2}}$

In this Appendix, we will use the moving plane method to show the uniqueness of the positive solution of the following nonlocal elliptic equation in $L^{\frac{2d}{d-2}}$

$$-\Delta\omega = \left(\frac{1}{|x|^4} * |\omega|^2 \right) \omega, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^d, d \geq 5. \quad (\text{A.1})$$

Proposition A.1. *The positive solution $\omega(x)$ of (A.1) in $L^{\frac{2d}{d-2}}(\mathbb{R}^d)$ is radially symmetry and decreasing about some point x_0 and unique. Therefore the positive solution $u(x)$ of (A.1) in $L^{\frac{2d}{d-2}}(\mathbb{R}^d)$ has the form*

$$c_0 \left(\frac{t}{t^2 + |x - x_0|^2} \right)^{\frac{d-2}{2}} \quad (\text{A.2})$$

with some positive constant c_0 and t .

Remark A.2. (1) The method of the moving planes was invented by Alexanderov in [1]. Later, it was further developed by Serrin [62], Gidas, Ni and Nirenberg [24], Caffarelli, Gidas and Spruck [5] in the study of the classification of semilinear elliptic equation $-\Delta\omega = \omega^{\frac{d+2}{d-2}}, x \in \mathbb{R}^d$. Subsequently, Chen and Li [11] and Li [40] simplified its proof. Recently, Wei and Xu [68] and Chen, Li and Ou [15] generalize the classification result to the solutions of higher order conformally invariant equations $(-\Delta)^{\alpha/2}\omega = \omega^{\frac{d+\alpha}{d-\alpha}}, x \in \mathbb{R}^d, 0 < \alpha < d$. Li [45] use the method of moving spheres to obtain the same classification result as that in [15]. For other applications, please refer to [10, 12, 13, 14, 19, 20, 48, 39].

- (2) The uniqueness still holds for the positive solution in $L_{loc}^{\frac{2d}{d-2}}(\mathbb{R}^d)$ by the analogue analysis as that in [15].
- (3) The results still hold for the fractional Laplacian equation ($0 < \alpha < d$)

$$(-\Delta)^{\alpha/2}\omega = \left(\frac{1}{|x|^{2\alpha}} * |\omega|^2 \right) \omega, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^d.$$

That is, the positive solution in $L_{loc}^{2d/(d-\alpha)}$ is radially symmetry and decreasing about some point x_0 and unique.

To do so, we first show the covariance of (A.1) under the Kelvin transform. Denote $K\omega$ the Kelvin transform of ω , that is

$$K\omega = \frac{1}{|x|^{n-2}} \omega \left(\frac{x}{|x|^2} \right).$$

Lemma A.3. *Let $\omega(x)$ be a solution of (A.1), then $w = K\omega$ is still a solution of (A.1).*

Proof. Note that

$$\omega(x) = \frac{1}{|x|^{d-2}} w \left(\frac{x}{|x|^2} \right),$$

It is well known that

$$\Delta\omega(x) = |x|^{-2-d}(\Delta w) \left(\frac{x}{|x|^2} \right).$$

Now we check the nonlinearity,

$$\begin{aligned} \left(\frac{1}{|\cdot|^4} * |\omega|^2 \right) (x) &= \int_{\mathbb{R}^d} \frac{|\omega(y)|^2}{|x-y|^4} dy = \int_{\mathbb{R}^d} \frac{|w(\frac{y}{|y|^2})|^2}{|x-y|^4 |y|^{2(d-2)}} dy \\ &= \int_{\mathbb{R}^d} \frac{|w(y)|^2}{|x - \frac{y}{|y|^2}|^4} |y|^{2(d-2)} |y|^{-2d} dy \\ &= |x|^{-4} \int_{\mathbb{R}^d} \frac{|w(y)|^2}{|\frac{x}{|x|^2} - y|^4} dy = |x|^{-4} \left(\frac{1}{|\cdot|^4} * |w|^2 \right) \left(\frac{x}{|x|^2} \right) \end{aligned}$$

where we use the following identity in the fourth step

$$|y|^4 \left| x - \frac{y}{|y|^2} \right|^4 = |x|^4 \left| \frac{x}{|x|^2} - y \right|^4.$$

Therefore, by (A.1), we have

$$|x|^{-2-d}(-\Delta w) \left(\frac{x}{|x|^2} \right) = |x|^{-4} \left(\frac{1}{|\cdot|^4} * |w|^2 \right) \left(\frac{x}{|x|^2} \right) \times \frac{1}{|x|^{d-2}} w \left(\frac{x}{|x|^2} \right).$$

This shows that w is also the solution of (A.1), that is, the equation (A.1) is covariant under the Kelvin transform. \square

Now, we transfer (A.1) into the equivalent integral system (A.3), then make use of the moving plane method in its global form to show that the positive solution of (A.1) in $L^{\frac{2d}{d-2}}$ is radially symmetric and monotone decreasing about some point $x_0 \in \mathbb{R}^d$. For this purpose, we first introduce some notations. For $x = (x_1, x_2, \dots, x_d) \in \mathbb{R}^d$, $\lambda \in \mathbb{R}$, we define $x^\lambda = (2\lambda - x_1, x_2, \dots, x_d)$, and

$$\omega_\lambda(x) = \omega(x^\lambda), \quad v_\lambda(x) = v(x^\lambda).$$

Let $\Sigma_\lambda = \{x = (x_1, x_2, \dots, x_d) \in \mathbb{R}^d, x_1 \geq \lambda\}$, we denote

$$\begin{aligned} \Sigma_\lambda^\omega &= \{x \in \Sigma_\lambda, \omega(x) < \omega_\lambda(x)\}; \quad \overline{\Sigma}_\lambda^\omega = \{x \in \Sigma_\lambda, \omega(x) \leq \omega_\lambda(x)\}, \\ \Sigma_\lambda^v &= \{x \in \Sigma_\lambda, v(x) < v_\lambda(x)\}. \end{aligned}$$

We denote the complement of Σ_λ in \mathbb{R}^d by Σ_λ^c , and the reflection of Σ_λ^ω about the plane $x_1 = \lambda$ by $(\Sigma_\lambda^\omega)^*$.

Let $v(x) = |x|^{-4} * |\omega|^2$, then (A.1) is equivalent to

$$\omega(x) = \int_{\mathbb{R}^d} \frac{\omega(y)v(y)}{|x-y|^{d-2}} dy, \quad v(x) = \int_{\mathbb{R}^d} \frac{\omega(y)^2}{|x-y|^4} dy. \quad (\text{A.3})$$

By $\omega \in L^{\frac{2d}{d-2}}$ and Lemma 2.2, we know that $v \in L^{d/2}$. Therefore, it suffices to show the radial symmetry and monotone decreasing of the positive solution (ω, v) of (A.3) in $L^{2d/(d-2)} \times L^{d/2}$ in order to show the radial symmetry and monotone decreasing of the positive solution ω of (A.1) in $L^{2d/(d-2)}$.

To do so, we decompose $\omega_\lambda(x)$, $\omega(x)$ in Σ_λ and $v_\lambda(x)$, $v(x)$ in Σ_λ as follows:

Lemma A.4. *For any solution (ω, v) of (A.3), we have*

$$\omega_\lambda(x) - \omega(x) = \int_{\Sigma_\lambda} \left(\frac{1}{|x - y|^{d-2}} - \frac{1}{|x^\lambda - y|^{d-2}} \right) (\omega_\lambda(y)v_\lambda(y) - \omega(y)v(y)) dy, \quad (\text{A.4})$$

$$v_\lambda(x) - v(x) = \int_{\Sigma_\lambda} \left(\frac{1}{|x - y|^4} - \frac{1}{|x^\lambda - y|^4} \right) (\omega_\lambda(y)^2 - \omega(y)^2) dy. \quad (\text{A.5})$$

Proof. By (A.3) and the fact that $|x - y^\lambda| = |x^\lambda - y|$, we have

$$\begin{aligned} \omega(x) &= \int_{\mathbb{R}^d} \frac{\omega(y)v(y)}{|x - y|^{d-2}} dy = \int_{\Sigma_\lambda} \frac{\omega(y)v(y)}{|x - y|^{d-2}} dy + \int_{\Sigma_\lambda^c} \frac{\omega(y)v(y)}{|x - y|^{d-2}} dy \\ &= \int_{\Sigma_\lambda} \frac{\omega(y)v(y)}{|x - y|^{d-2}} dy + \int_{\Sigma_\lambda} \frac{\omega(y^\lambda)v(y^\lambda)}{|x - y^\lambda|^{d-2}} dy \\ &= \int_{\Sigma_\lambda} \frac{\omega(y)v(y)}{|x - y|^{d-2}} dy + \int_{\Sigma_\lambda} \frac{\omega(y^\lambda)v(y^\lambda)}{|x^\lambda - y|^{d-2}} dy \\ &= \int_{\Sigma_\lambda} \left(\frac{\omega(y)v(y)}{|x - y|^{d-2}} + \frac{\omega_\lambda(y)v_\lambda(y)}{|x^\lambda - y|^{d-2}} \right) dy. \end{aligned} \quad (\text{A.6})$$

This implies that

$$\omega_\lambda(x) = \omega(x^\lambda) = \int_{\Sigma_\lambda} \left(\frac{\omega(y)v(y)}{|x^\lambda - y|^{d-2}} + \frac{\omega_\lambda(y)v_\lambda(y)}{|x - y|^{d-2}} \right) dy. \quad (\text{A.7})$$

By (A.6) and (A.7), we get (A.4). By the same way, we can show (A.5). \square

Based on the above preliminaries, we can prove that

Proposition A.5. *Let (ω, v) be the positive solution of (A.3) in $L^{2d/(d-2)} \times L^{d/2}$. Then ω and v are radially symmetry and decreasing about some point $x_0 \in \mathbb{R}^d$.*

Proof. The proof consist of three steps.

Step 1: We show that there exists an $N > 0$ such that for any $\lambda < -N$, we have

$$\omega(x) \geq \omega_\lambda(x); \text{ and } v(x) \geq v_\lambda(x), \forall x \in \Sigma_\lambda. \quad (\text{A.8})$$

Step 2: We move the plane continuously from $\lambda < -N$ to the right as long as (A.8) holds. We show that if the plane stops at $x_1 = \lambda_0$ for some λ_0 , then $\omega(x)$ and $v(x)$ must be symmetric and monotone about the plane $x_1 = \lambda_0$; otherwise, we can move the plane all the way to the right.

Step 3: By Step 1, we know that the plane cannot move all through to the right in Step 2. That is, the plane will eventually stop at some finite point. In fact, by the similar analysis as that in Step 1 and Step 2, there exists a large M , such that for $\lambda > M$, we have

$$\omega(x) \leq \omega_\lambda(x); \text{ and } v(x) \leq v_\lambda(x), \forall x \in \Sigma_\lambda.$$

Now we can move the plane continuously from $\lambda > M$ to the left as long the above fact holds. The planes moved from the left and the right will eventually meet at some point. Finally, since the direction of x_1 can be chosen arbitrarily, we deduce that $\omega(x)$ and $v(x)$ must be radially symmetric and decreasing about some point.

According to the above analysis, we only need to show the facts in Step 1 and Step 2.

Step 1: For the sufficiently negative value of λ , we can show that Σ_λ^ω and Σ_λ^v must be both empty. In fact, for any $x \in \Sigma_\lambda$, we have

$$\begin{aligned}
\omega_\lambda(x) - \omega(x) &= \int_{\Sigma_\lambda} \left(\frac{1}{|x-y|^{d-2}} - \frac{1}{|x^\lambda-y|^{d-2}} \right) (\omega_\lambda(y)v_\lambda(y) - \omega(y)v(y)) dy \\
&\leq \int_{\{y \in \Sigma_\lambda: \omega v \leq \omega_\lambda v_\lambda\}} \frac{1}{|x-y|^{d-2}} (\omega_\lambda(y)v_\lambda(y) - \omega(y)v(y)) dy \\
&= \int_{\{y \in \Sigma_\lambda: \omega v \leq \omega_\lambda v_\lambda\}} \frac{\omega(y)(v_\lambda(y) - v(y)) + v_\lambda(y)(\omega_\lambda(y) - \omega(y))}{|x-y|^{d-2}} dy \\
&= \int_{\{y \in \Sigma_\lambda: \omega v \leq \omega_\lambda v_\lambda, \omega < \omega_\lambda, v < v_\lambda\}} \frac{\omega(y)(v_\lambda(y) - v(y)) + v_\lambda(y)(\omega_\lambda(y) - \omega(y))}{|x-y|^{d-2}} dy \\
&\quad + \int_{\{y \in \Sigma_\lambda: \omega v \leq \omega_\lambda v_\lambda, \omega < \omega_\lambda, v \geq v_\lambda\}} \frac{\omega(y)(v_\lambda(y) - v(y)) + v_\lambda(y)(\omega_\lambda(y) - \omega(y))}{|x-y|^{d-2}} dy \\
&\quad + \int_{\{y \in \Sigma_\lambda: \omega v \leq \omega_\lambda v_\lambda, \omega \geq \omega_\lambda, v < v_\lambda\}} \frac{\omega(y)(v_\lambda(y) - v(y)) + v_\lambda(y)(\omega_\lambda(y) - \omega(y))}{|x-y|^{d-2}} dy \\
&\leq \int_{\Sigma_\lambda^v} \frac{\omega(y)}{|x-y|^{d-2}} (v_\lambda(y) - v(y)) dy + \int_{\Sigma_\lambda^\omega} \frac{v_\lambda(y)}{|x-y|^{d-2}} (\omega_\lambda(y) - \omega(y)) dy.
\end{aligned}$$

Hence, by Lemma 2.2 and Hölder's inequality, we obtain

$$\begin{aligned}
\|\omega_\lambda - \omega\|_{L^{\frac{2d}{d-2}}(\Sigma_\lambda^\omega)} &\leq C\|\omega(v_\lambda - v)\|_{L^{\frac{2d}{d+2}}(\Sigma_\lambda^v)} + C\|v_\lambda(\omega_\lambda - \omega)\|_{L^{\frac{2d}{d+2}}(\Sigma_\lambda^\omega)} \\
&\leq C\|\omega\|_{L^{\frac{2d}{d-2}}(\Sigma_\lambda^v)}\|v_\lambda - v\|_{L^{\frac{d}{2}}(\Sigma_\lambda^v)} + C\|v_\lambda\|_{L^{\frac{d}{2}}(\Sigma_\lambda^\omega)}\|\omega_\lambda - \omega\|_{L^{\frac{2d}{d-2}}(\Sigma_\lambda^\omega)}. \quad (\text{A.9})
\end{aligned}$$

In the same argument as above, for $x \in \Sigma_\lambda$, we also have

$$v_\lambda(x) - v(x) \leq 2 \int_{\Sigma_\lambda^\omega} \frac{\omega_\lambda(y)}{|x-y|^4} (\omega_\lambda(y) - \omega(y)) dy,$$

and

$$\|v_\lambda - v\|_{L^{\frac{d}{2}}(\Sigma_\lambda^v)} \leq C\|\omega_\lambda(\omega_\lambda - \omega)\|_{L^{\frac{d}{d-2}}(\Sigma_\lambda^\omega)} \leq C\|\omega_\lambda\|_{L^{\frac{2d}{d-2}}(\Sigma_\lambda^\omega)}\|\omega_\lambda - \omega\|_{L^{\frac{2d}{d-2}}(\Sigma_\lambda^\omega)}. \quad (\text{A.10})$$

Hence, taking (A.10) into (A.9), we obtain

$$\begin{aligned}
&\|\omega_\lambda - \omega\|_{L^{\frac{2d}{d-2}}(\Sigma_\lambda^\omega)} \\
&\leq C\|\omega\|_{L^{\frac{2d}{d-2}}(\Sigma_\lambda^v)}\|\omega_\lambda\|_{L^{\frac{2d}{d-2}}(\Sigma_\lambda^\omega)}\|\omega_\lambda - \omega\|_{L^{\frac{2d}{d-2}}(\Sigma_\lambda^\omega)} + C\|v_\lambda\|_{L^{\frac{d}{2}}(\Sigma_\lambda^\omega)}\|\omega_\lambda - \omega\|_{L^{\frac{2d}{d-2}}(\Sigma_\lambda^\omega)} \\
&\leq C\|\omega\|_{L^{\frac{2d}{d-2}}(\Sigma_\lambda)}\|\omega_\lambda\|_{L^{\frac{2d}{d-2}}(\Sigma_\lambda)}\|\omega_\lambda - \omega\|_{L^{\frac{2d}{d-2}}(\Sigma_\lambda^\omega)} + C\|v_\lambda\|_{L^{\frac{d}{2}}(\Sigma_\lambda)}\|\omega_\lambda - \omega\|_{L^{\frac{2d}{d-2}}(\Sigma_\lambda^\omega)} \\
&\leq \left(C\|\omega\|_{L^{\frac{2d}{d-2}}(\Sigma_\lambda)}\|\omega\|_{L^{\frac{2d}{d-2}}(\Sigma_\lambda^c)} + C\|v\|_{L^{\frac{d}{2}}(\Sigma_\lambda^c)} \right) \|\omega_\lambda - \omega\|_{L^{\frac{2d}{d-2}}(\Sigma_\lambda^\omega)}. \quad (\text{A.11})
\end{aligned}$$

By $\omega \in L^{2d/(d-2)}$ and $v \in L^{d/2}$ and the dominated convergence theorem, we can choose N sufficiently large such that $\lambda < -N$, and

$$C\|\omega\|_{L^{\frac{2d}{d-2}}(\Sigma_\lambda)}\|\omega\|_{L^{\frac{2d}{d-2}}(\Sigma_\lambda^c)} + C\|v\|_{L^{\frac{d}{2}}(\Sigma_\lambda^c)} \leq \frac{1}{2}.$$

Thus (A.11) implies that

$$\|\omega_\lambda - \omega\|_{L^{\frac{2d}{d-2}}(\Sigma_\lambda^\omega)} = 0.$$

This implies that Σ_λ^ω must be the set with zero measure, hence must be empty, up to a set with zero measure. By (A.10), Σ_λ^v also must be empty.

Step 2: We move the plane $x_1 = \lambda$ to the right as long as (A.8) holds. Suppose that at some λ_0 , we have

$$\begin{aligned} \omega(x) &\geq \omega_{\lambda_0}(x), v(x) \geq v_{\lambda_0}(x), \text{ on } \Sigma_{\lambda_0}, \\ \text{but } \omega(x) &\not\equiv \omega_{\lambda_0}(x) \text{ or } v(x) \not\equiv v_{\lambda_0}(x) \text{ on } \Sigma_{\lambda_0}. \end{aligned}$$

We show that the plane can be moved further to the right. More precisely, there exists $\epsilon = \epsilon(n, \omega, v)$ such that $\omega(x) \geq \omega_\lambda(x)$ and $v(x) \geq v_\lambda(x)$ on Σ_λ for all $\lambda \in [\lambda_0, \lambda_0 + \epsilon)$.

In the case

$$v(x) \not\equiv v_{\lambda_0}(x) \text{ on } \Sigma_{\lambda_0}.$$

By (A.4), we have that $\omega(x) > \omega_{\lambda_0}(x)$ in the interior of Σ_{λ_0} . Note that

$$|\overline{\Sigma_{\lambda_0}^\omega}| = 0, \text{ and } \lim_{\lambda \rightarrow \lambda_0} \Sigma_\lambda^\omega \subseteq \overline{\Sigma_{\lambda_0}^\omega}.$$

From (A.9) and (A.10), we have

$$\|\omega_\lambda - \omega\|_{L^{\frac{2d}{d-2}}(\Sigma_\lambda^\omega)} \leq \left(C\|\omega\|_{L^{\frac{2d}{d-2}}(\Sigma_\lambda)}\|\omega\|_{L^{\frac{2d}{d-2}}((\Sigma_\lambda^\omega)^*)} + C\|v\|_{L^{\frac{d}{2}}((\Sigma_\lambda^\omega)^*)} \right) \|\omega_\lambda - \omega\|_{L^{\frac{2d}{d-2}}(\Sigma_\lambda^\omega)}.$$

By $\omega \in L^{2d/(d-2)}$ and $v \in L^{d/2}$ and $|\overline{\Sigma_{\lambda_0}^\omega}| = 0$ and the dominated convergence theorem, we can choose ϵ sufficiently small, such that for all $\lambda \in [\lambda_0, \lambda_0 + \epsilon)$, we have

$$C\|\omega\|_{L^{\frac{2d}{d-2}}(\Sigma_\lambda)}\|\omega\|_{L^{\frac{2d}{d-2}}((\Sigma_\lambda^\omega)^*)} + C\|v\|_{L^{\frac{d}{2}}((\Sigma_\lambda^\omega)^*)} \leq \frac{1}{2}.$$

This implies that $\|\omega_\lambda - \omega\|_{L^{\frac{2d}{d-2}}(\Sigma_\lambda^\omega)} = 0$. Hence, Σ_λ^ω must be empty for all $\lambda \in [\lambda_0, \lambda_0 + \epsilon)$. It also implies that Σ_λ^v also is empty for all $\lambda \in [\lambda_0, \lambda_0 + \epsilon)$.

In the case

$$\omega(x) \not\equiv \omega_{\lambda_0}(x) \text{ on } \Sigma_{\lambda_0}.$$

By (A.5), we have that $v(x) > v_{\lambda_0}(x)$ in the interior of Σ_{λ_0} . By the above analysis, we know that Σ_λ^ω and Σ_λ^v must also be empty for all $\lambda \in [\lambda_0, \lambda_0 + \epsilon)$. This completes the proof. \square

Now we use the elliptic regularity theory to show that

Proposition A.6. *Assume that ω is a positive solution of (A.1) in $L^{2d/(d-2)}$. Then ω is uniformly bounded in \mathbb{R}^d . Furthermore, ω is C^∞ and*

$$\lim_{|x| \rightarrow +\infty} |x|^{d-2} \omega(x) = \omega_\infty \tag{A.12}$$

for some positive number ω_∞ .

Proof. Step 1: We first show that ω is uniformly bounded and continuous. For $A > 0$, we define

$$\Omega = \{x \in \mathbb{R}^n, \omega(x) > A\} \quad \text{and} \quad \omega_A(x) = \begin{cases} \omega(x), & \Omega, \\ 0, & \mathbb{R}^d \setminus \Omega. \end{cases}$$

Hence

$$\omega - \omega_A \in L^{2d/(d-2)} \cap L^\infty, \quad \text{for any } A > 0. \quad (\text{A.13})$$

Since ω is a solution of (A.1), we have

$$\omega(x) = \int_{\mathbb{R}^d} \frac{(|\cdot|^{-4} * |\omega|^2)\omega(y)}{|x-y|^{d-2}} dy, \quad \forall x \in \mathbb{R}^d.$$

This implies that for any $x \in \Omega$

$$\begin{aligned} \omega_A(x) &= \int_{\mathbb{R}^d} \frac{(|\cdot|^{-4} * |\omega|^2)\omega(y)}{|x-y|^{d-2}} dy, \\ &= \int_{\mathbb{R}^d} \frac{(|\cdot|^{-4} * |\omega_A|^2)\omega_A(y)}{|x-y|^{d-2}} dy + \int_{\mathbb{R}^d} \frac{(|\cdot|^{-4} * |\omega - \omega_A|^2)\omega_A(y)}{|x-y|^{d-2}} dy \\ &\quad + \int_{\mathbb{R}^d} \frac{(|\cdot|^{-4} * |\omega_A|^2)(\omega - \omega_A)(y)}{|x-y|^{d-2}} dy + \int_{\mathbb{R}^d} \frac{(|\cdot|^{-4} * |\omega - \omega_A|^2)(\omega - \omega_A)(y)}{|x-y|^{d-2}} dy. \end{aligned}$$

For any $r \geq \frac{2d}{d-2}$, we have by Lemma 2.2

$$\begin{aligned} \|\omega_A\|_{L^r} &\leq C \|\omega_A\|_{L^{2d/(d-2)}}^2 \|\omega_A\|_{L^r} + C \|\omega_A\|_{L^{2d/(d-2)}} \|\omega - \omega_A\|_{L^{2d/(d-2)}} \|\omega - \omega_A\|_{L^r} \\ &\quad + C \|\omega_A\|_{L^{2d/(d-2)}}^2 \|\omega - \omega_A\|_{L^r} + C \|\omega - \omega_A\|_{L^{2d/(d-2)}}^2 \|\omega - \omega_A\|_{L^r}. \end{aligned} \quad (\text{A.14})$$

On one hand, from $\omega \in L^{2d/(d-2)}$, we can choose A sufficiently large, such that

$$C \|\omega_A\|_{L^{2d/(d-2)}}^2 \leq \frac{1}{2}. \quad (\text{A.15})$$

On the other hand, by $\omega \in L^{2d/(d-2)}$ and (A.13), we easily verify that

$$\|\omega_A\|_{L^{2d/(d-2)}} + \|\omega - \omega_A\|_{L^{2d/(d-2)}} + \|\omega - \omega_A\|_{L^r} \leq C(A). \quad (\text{A.16})$$

Taking (A.15) and (A.16) into (A.14), we have for any $r \geq \frac{2d}{d-2}$

$$\|\omega_A\|_{L^r} \leq \frac{1}{2} \|\omega_A\|_{L^r} + C(A),$$

which implies that $\omega_A \in L^r$ for any $r \geq \frac{2d}{d-2}$. Therefore, we have $\omega \in L^r$ for any $r \geq \frac{2d}{d-2}$. By Lemma 2.2, we have

$$-\Delta\omega = (|\cdot|^{-4} * |\omega|^2)\omega \in L^p, \quad \text{for any } p \geq \frac{2d}{d+2}.$$

From the L^p theory and the Sobolev embedding theorem in [63], we know that ω is uniformly bounded and $C^{0,\gamma}$ for all $0 < \gamma < 1$. In fact, we also have $\omega \in C^\infty$ from Theorem 4.4.8 in [8].

Step 2: We show that ω has the asymptotic behavior at infinity. We prove it by contradiction. Consider the Kelvin transform solution:

$$U(x) = \frac{1}{|x|^{d-2}} \omega\left(\frac{x}{|x|^2}\right) \implies |x|^{d-2} \omega(x) = U\left(\frac{x}{|x|^2}\right).$$

Applying Proposition A.5 to $U(x)$, we conclude that $U(x)$ must be radially symmetric about some point and continuity. Hence

$$\lim_{|x| \rightarrow +\infty} |x|^{d-2} \omega(x) = U(0) > 0.$$

This completes the proof of Proposition A.6. \square

Finally, the uniqueness in Proposition A.1 comes from the scaling covariance of (A.1) and the uniqueness of ODE theory. This finishes the proof of Proposition A.1. \square

APPENDIX B. COERCIVITY OF Φ ON $H^\perp \cap \dot{H}_{rad}^1$

In this Appendix, we prove Proposition 2.14. We divide the proof into two steps.

Step 1: Nonnegative. We claim that for any function $h \in \dot{H}^1$, $(h, W)_{\dot{H}^1} = 0$, there exists

$$\Phi(h) \geq 0.$$

Indeed, let

$$I(u) = \frac{\|\nabla u\|_2^4}{\|\nabla W\|_2^4} - \frac{\iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|u(x)|^2 |u(y)|^2}{|x-y|^4} dx dy}{\iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|W(x)|^2 |W(y)|^2}{|x-y|^4} dx dy}. \quad (\text{B.1})$$

By Lemma 2.6, we have

$$\forall u \in \dot{H}^1, \quad I(u) \geq 0.$$

Choosing $h \in \dot{H}^1$ such that $(W, h)_{\dot{H}^1} = 0$, $\alpha \in \mathbb{R}$, we consider the expansion of $I(W + \alpha h)$ in α of order 2. Note that

$$\|\nabla(W + \alpha h)\|_2^4 = \|\nabla W\|_2^4 \left(1 + 2\alpha^2 \frac{\|h\|_{\dot{H}^1}^2}{\|W\|_{\dot{H}^1}^2} + O(\alpha^4) \right),$$

and

$$\begin{aligned} & \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|W(x) + \alpha h(x)|^2 |W(y) + \alpha h(y)|^2}{|x-y|^4} dx dy \\ &= \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|W(x)|^2 |W(y)|^2}{|x-y|^4} dx dy \\ & \quad + 2\alpha \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|W(x)|^2 W(y) h_1(y) + |W(y)|^2 W(x) h_1(x)}{|x-y|^4} dx dy \\ & \quad + \alpha^2 \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|W(x)|^2 |h(y)|^2 + |W(y)|^2 |h(x)|^2 + 4W(x) h_1(x) W(y) h_1(y)}{|x-y|^4} dx dy + O(\alpha^3) \\ &= \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|W(x)|^2 |W(y)|^2}{|x-y|^4} dx dy \\ & \quad \times \left(1 + 2\alpha^2 \frac{\iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|W(x)|^2 |h(y)|^2 + 2W(x) h_1(x) W(y) h_1(y)}{|x-y|^4} dx dy}{\|W\|_{\dot{H}^1}^2} + O(\alpha^3) \right), \end{aligned}$$

one easily shows that

$$I(W + \alpha h) = \frac{2\alpha^2}{\|W\|_{\dot{H}^1}^2} \left(\|h\|_{\dot{H}^1}^2 - \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|W(x)|^2 |h(y)|^2 + 2W(x)h_1(x)W(y)h_1(y)}{|x-y|^4} dx dy \right) + O(\alpha^3)$$

where we have used the facts that

$$\begin{aligned} \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|W(x)|^2 W(y)h_1(y)}{|x-y|^4} dx dy &= \int_{\mathbb{R}^d} -\Delta W(y) \cdot h_1(y) dy = 0, \\ \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|W(y)|^2 W(x)h_1(x)}{|x-y|^4} dx dy &= \int_{\mathbb{R}^d} -\Delta W(x) \cdot h_1(x) dy = 0, \\ \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|W(x)|^2 |h(y)|^2}{|x-y|^4} dx dy &= \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|W(y)|^2 |h(x)|^2}{|x-y|^4} dx dy, \\ \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|W(x)|^2 |W(y)|^2}{|x-y|^4} dx dy &= \int_{\mathbb{R}^d} |\nabla W|^2 dx = C_*^{-4}. \end{aligned}$$

Since $I(W + \alpha h) \geq 0$ for all $\alpha \in \mathbb{R}$, we have

$$\Phi(h) = \int_{\mathbb{R}^d} |\nabla h|^2 dx - \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|W(x)|^2 |h(y)|^2 + 2W(x)h_1(x)W(y)h_1(y)}{|x-y|^4} dx dy \geq 0.$$

Step 2: Coercivity. We show that there exists a constant $c_* > 0$ such that for any radial function $h \in H^\perp$

$$\Phi(h) \geq c_* \|h\|_{\dot{H}^1}^2.$$

We rewrite $\Phi(h) = \Phi_1(h_1) + \Phi_2(h_2)$, where

$$\begin{aligned} \Phi_1(h_1) &:= \frac{1}{2} \int_{\mathbb{R}^d} (L_+ h_1) h_1 dx \\ &= \frac{1}{2} \int_{\mathbb{R}^d} |\nabla h_1|^2 dx - \frac{1}{2} \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|W(x)|^2 |h_1(y)|^2}{|x-y|^4} dx dy \\ &\quad - \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{W(x)h_1(x)W(y)h_1(y)}{|x-y|^4} dx dy, \end{aligned}$$

and

$$\begin{aligned} \Phi_2(h_2) &:= \frac{1}{2} \int_{\mathbb{R}^d} (L_- h_2) h_2 dx \\ &= \frac{1}{2} \int_{\mathbb{R}^d} |\nabla h_2|^2 dx - \frac{1}{2} \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|W(x)|^2 |h_2(y)|^2}{|x-y|^4} dx dy. \end{aligned}$$

By Step 1, L_+ is nonnegative on $\{W\}^\perp$ (in the sense of \dot{H}^1) and L_- is nonnegative. We will deduce the coercivity by the compactness argument [69, Proposition 2.9].

We first show that there exists a constant c such that

$$\Phi_1(h_1) \geq c \|h_1\|_{\dot{H}^1}^2,$$

for any radial real-valued \dot{H}^1 -function $h_1 \in \{W, \widetilde{W}\}^\perp$ (in the sense of \dot{H}^1). Assume that the above inequality does not hold, then there exists a sequence of real-valued radial \dot{H}^1 -functions

$\{f_n\}_n$ such that

$$f_n \in H^\perp, \quad \lim_{n \rightarrow +\infty} \Phi_1(f_n) = 0, \quad \|f_n\|_{\dot{H}^1} = 1. \quad (\text{B.2})$$

Extracting a subsequence from (f_n) , we may assume that

$$f_n \rightharpoonup f_* \text{ in } \dot{H}^1.$$

The weak convergence of $f_n \in H^\perp$ to f_* implies that $f_* \in H^\perp$. In addition, by compactness, we have

$$\begin{aligned} \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|W(x)|^2 |f_n(y)|^2}{|x-y|^4} dx dy &\longrightarrow \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|W(x)|^2 |f_*(y)|^2}{|x-y|^4} dx dy, \\ \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{W(x) f_n(x) W(y) f_n(y)}{|x-y|^4} dx dy &\longrightarrow \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{W(x) f_*(x) W(y) f_*(y)}{|x-y|^4} dx dy. \end{aligned}$$

Thus by the Fatou lemma, (B.2) and Step 1, we have

$$0 \leq \Phi_1(f_*) \leq \liminf_{n \rightarrow +\infty} \Phi_1(f_n) = 0.$$

So we conclude that f_* is the solution to the following minimizing problem

$$0 = \min_{f \in \Omega \setminus \{0\}} \frac{\int_{\mathbb{R}^d} L_+ f \cdot f dx}{\|f\|_{\dot{H}^1}^2}, \quad \Omega = \left\{ f \in \dot{H}_{rad}^1, (f, W)_{\dot{H}^1} = (f, \widetilde{W})_{\dot{H}^1} = 0 \right\}.$$

Thus for some Langrange multipliers λ_0, λ_1 , we have

$$L_+ f_* = \lambda_0 \Delta W + \lambda_1 \widetilde{\Delta W}.$$

Note that $(W, \widetilde{W})_{\dot{H}^1} = 0$ and $L_+(\widetilde{W}) = 0$, we have

$$0 = \int_{\mathbb{R}^d} f_* L_+(\widetilde{W}) = \int_{\mathbb{R}^d} (L_+ f_*) \widetilde{W} = \lambda_1 \|\widetilde{W}\|_{\dot{H}^1}^2 \implies \lambda_1 = 0.$$

This tells us that

$$L_+ f_* = \lambda_0 \Delta W = -\lambda_0 (|x|^{-4} * |W|^2) W = \frac{\lambda_0}{2} L_+ W.$$

By $\text{Null}(L_+) = \text{span}\{\widetilde{W}\}$ in Lemma 2.13, there exists μ_1 such that

$$f_* = \frac{\lambda_0}{2} W + \mu_1 \widetilde{W}.$$

Using $f_* \in H^\perp$, we get $\mu_1 = 0$ and

$$f_* = \frac{\lambda_0}{2} W.$$

This implies that

$$0 = \Phi_1(f_*) = \frac{\lambda_0^2}{4} \Phi_1(W) = \frac{\lambda_0^2}{4} \Phi(W) \leq 0 \implies \lambda_0 = 0.$$

Therefore, we have

$$f_* = 0, \quad \text{and} \quad f_n \rightharpoonup 0 \text{ in } \dot{H}^1.$$

Now, by compactness, we have

$$\iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{|W(x)|^2 |f_n(y)|^2}{|x-y|^4} \longrightarrow 0, \quad \iint_{\mathbb{R}^d \times \mathbb{R}^d} \frac{W(x) f_n(x) W(y) f_n(y)}{|x-y|^4} \longrightarrow 0.$$

By $\Phi_1(f_n) \rightarrow 0$ in (B.2), we get that

$$\|\nabla f_n\|_2 \rightarrow 0,$$

which contradicts $\|f_n\|_{\dot{H}^1} = 1$ in (B.2).

Using the same argument, we can show that there exists a constant c , such that for any real-valued radial \dot{H}^1 -function $h_2 \in \{W\}^\perp$, we have

$$\Phi_2(h_2) \geq c\|h_2\|_{\dot{H}^1}^2.$$

This completes the proof of Proposition 2.14. \square

APPENDIX C. SPECTRAL PROPERTIES OF THE LINEARIZED OPERATOR

In this Appendix, we will give the proof of Proposition 2.15.

C.1. Existence, symmetry of the eigenfunctions. Note that

$$\overline{\mathcal{L}v} = -\mathcal{L}\overline{v},$$

so that if $e_0 > 0$ is an eigenvalue of \mathcal{L} with the eigenfunction \mathcal{Y}_+ , $-e_0$ is also an eigenvalue with eigenfunction $\mathcal{Y}_- = \overline{\mathcal{Y}_+}$.

Now we show the existence of \mathcal{Y}_+ . Let $\mathcal{Y}_1 = \Re \mathcal{Y}_+$, $\mathcal{Y}_2 = \Im \mathcal{Y}_+$, it suffices to show that

$$-L_- \mathcal{Y}_2 = e_0 \mathcal{Y}_1, \quad L_+ \mathcal{Y}_1 = e_0 \mathcal{Y}_2. \quad (\text{C.1})$$

From the proof of the coercivity property of Φ on $H^\perp \cap \dot{H}_{rad}^1$ in Proposition 2.14, we know that L_- on L^2 with domain H^2 is self-adjoint and nonnegative, By Theorem 3.35 in [29, Page 281], it has a unique square root $(L_-)^{1/2}$ with domain H^1 .

Assume that there exists a function $f \in H_{rad}^4$ such that

$$\mathcal{P}f = -e_0^2 f_1, \quad \mathcal{P} := (L_-)^{1/2} (L_+) (L_-)^{1/2}. \quad (\text{C.2})$$

Then taking

$$\mathcal{Y}_1 := (L_-)^{1/2} f, \quad \mathcal{Y}_2 := \frac{1}{e_0} (L_+) (L_-)^{1/2} f$$

would yield a solution of (C.1), which implies the existence of the radial \mathcal{Y}_\pm by the rotation invariance of the operator \mathcal{L} .

It suffices to show that the operator \mathcal{P} on L^2 with domain H_{rad}^4 has a strictly negative eigenvalue. Since \mathcal{P} is a relatively compact, self-adjoint, perturbation of $(-\Delta)^2$, then by the Weyl theorem [27, 29], we know that

$$\sigma_{ess}(\mathcal{P}) = [0, +\infty).$$

We only need to show that \mathcal{P} has at least one negative eigenvalue $-e_0^2$.

Lemma C.1.

$$\sigma_-(\mathcal{P}) := \inf\{(\mathcal{P}f, f)_{L^2}, f \in H_{rad}^4, \|f\|_{L^2} = 1\} < 0.$$

Proof. Note that

$$(\mathcal{P}f, f)_{L^2} = (L_+ F, F)_{L^2}, \quad F := (L_-)^{1/2} f,$$

it suffices to find F such that there exists $g \in H_{rad}^4$, $F = (\Delta + V)g$ and

$$(L_+ F, F)_{L^2} < 0. \quad (\text{C.3})$$

Since $W \in L^2(\mathbb{R}^d)$, we have

$$\text{Ran}(L_-)^\perp = \text{Null}(L_-) = \text{span}\{W\},$$

Thus

$$\text{Ran}(L_-) = \{f \in L^2, (f, W)_{L^2} = 0\}. \quad (\text{C.4})$$

Note that L_+ is a self-adjoint compact perturbation of $-\Delta$ and

$$(L_+ W, W)_{L^2} = -2 \int |\nabla W|^2 dx < 0,$$

we easily see that L_+ has a negative eigenvalue. Let Z be the eigenfunction for this eigenvalue (it is radial by the minimax principle). Note that $L_+ \widetilde{W} = 0$, then for any real number α , we have

$$E_0 := \int_{\mathbb{R}^d} L_+(Z + \alpha \widetilde{W}) \cdot (Z + \alpha \widetilde{W}) = \int_{\mathbb{R}^d} L_+ Z \cdot Z < 0. \quad (\text{C.5})$$

Since

$$(\widetilde{W}, W)_{L^2} \neq 0,$$

we can choose the real number α_1 such that

$$(Z + \alpha_1 \widetilde{W}, W)_{L^2} = 0,$$

which means that

$$\begin{aligned} ((L_- + 1)(Z + \alpha_1 \widetilde{W}), W)_{L^2} &= (Z + \alpha_1 \widetilde{W}, (L_- + 1)W)_{L^2} \\ &= (Z + \alpha_1 \widetilde{W}, W)_{L^2} = 0. \end{aligned}$$

By (C.4), for any $\epsilon > 0$, there exists a function $G_\epsilon \in H_{rad}^2$ such that

$$\|L_- G_\epsilon - (L_- + 1)(Z + \alpha_1 \widetilde{W})\|_{L^2} < \epsilon.$$

Taking

$$F_\epsilon := (L_- + 1)^{-1} L_- G_\epsilon,$$

we obtain that

$$\|(L_- + 1)(F_\epsilon - (Z + \alpha_1 \widetilde{W}))\|_{L^2} \leq \epsilon.$$

This implies that

$$\|F_\epsilon - (Z + \alpha_1 \widetilde{W})\|_{H^2} \leq \epsilon \|(L_- + 1)^{-1}\|_{L^2 \rightarrow L^2}.$$

Hence for some constant C_0 , we have

$$\left| \int_{\mathbb{R}^d} L_+ F_\epsilon \cdot F_\epsilon - \int_{\mathbb{R}^d} L_+(Z + \alpha_1 \widetilde{W}) \cdot (Z + \alpha_1 \widetilde{W}) \right| \leq C_0 \epsilon.$$

By (C.5), we have (C.3) for $F = F_\epsilon$, $\epsilon = \frac{E_0}{2C_0}$. \square

C.2. Decay of the eigenfunctions at infinity. By the bootstrap argument, we know that $\mathcal{Y}_\pm \in C^\infty \cap H^\infty$. In fact, we have $\mathcal{Y}_\pm \in \mathcal{S}$. By (C.1), it suffices to show that $\mathcal{Y}_1 \in \mathcal{S}$. Note that \mathcal{Y}_1 satisfies that

$$\begin{aligned} (e_0^2 + \Delta^2) \mathcal{Y}_1 &= -2\Delta \left(\frac{1}{|x|^4} * (W \mathcal{Y}_1) \cdot W \right) - 2 \left(\frac{1}{|x|^4} * (W \mathcal{Y}_1) \right) \left(\frac{1}{|x|^4} * |W|^2 \right) \cdot W \\ &\quad - \Delta \left(\frac{1}{|x|^4} * |W|^2 \cdot \mathcal{Y}_1 \right) - \left(\frac{1}{|x|^4} * |W|^2 \cdot \Delta \mathcal{Y}_1 \right) - \left(\frac{1}{|x|^4} * |W|^2 \right)^2 \cdot \mathcal{Y}_1. \end{aligned}$$

Thus,

$$(e_0 - \Delta)^2 \mathcal{Y}_1 = -\Delta \left(\frac{1}{|x|^4} * |W|^2 \cdot \mathcal{Y}_1 \right) - \left(\frac{1}{|x|^4} * |W|^2 \cdot \Delta \mathcal{Y}_1 \right) - \left(\frac{1}{|x|^4} * |W|^2 \right)^2 \cdot \mathcal{Y}_1 - 2e_0 \Delta \mathcal{Y}_1$$

$$-2\Delta \left(\frac{1}{|x|^4} * (W\mathcal{Y}_1) \cdot W \right) - 2 \left(\frac{1}{|x|^4} * (W\mathcal{Y}_1) \right) \left(\frac{1}{|x|^4} * |W|^2 \right) \cdot W.$$

Because of the existence of the nonlocal interaction on the right hand side, the decay estimate in [16] does not work. From the Bessel potential theory in [63], we know that the integral kernel G of the operator $(e_0 - \Delta)^{-2}$ is

$$G(x) = \frac{1}{(4\pi)^2} \int_0^\infty e^{-\frac{e_0}{4\pi}\delta} e^{-\frac{\pi}{\delta}|x|^2} \delta^{-\frac{d+4}{2}} \frac{d\delta}{\delta}.$$

Hence we have

- (1) $G(x) = \frac{|x|^{-d+4}}{\gamma(4)} + o(|x|^{-d+4})$, $|x| \rightarrow 0$;
- (2) there exists $c > 0$ such that

$$G(x) = o(e^{-c|x|}), \quad |x| \rightarrow +\infty.$$

Then the conclusion follows by the analogue estimates in [21, 36]. \square

C.3. Coercivity of Φ on $G_\perp \cap \dot{H}_{rad}^1$. Let $f \in G_\perp \cap \dot{H}_{rad}^1$, we now decompose $f, \mathcal{Y}_+, \mathcal{Y}_-$ in the orthogonal sum $\dot{H}^1 = H \oplus H^\perp$:

$$\begin{aligned} f &= \alpha W + \tilde{h}, \quad \mathcal{Y}_+ = \eta i W + \xi \tilde{W} + \zeta W + h_+, \\ \mathcal{Y}_- &= -\eta i W + \xi \tilde{W} + \zeta W + h_-, \end{aligned}$$

where $\tilde{h}, h_+, h_- \in H^\perp \cap \dot{H}_{rad}^1$ and $h_- = \bar{h}_+$.

Step 1. We first show that for any $f \in G_\perp$,

$$\Phi(f) = -\frac{B(h_+, \tilde{h})B(h_-, \tilde{h})}{\sqrt{\Phi(h_+)}\sqrt{\Phi(h_-)}} + \Phi(\tilde{h}) \quad (\text{C.6})$$

By $\Phi(\mathcal{Y}_\pm) = 0$ and Remark 2.16, we have

$$\zeta^2 \Phi(W) + \Phi(h_+) = 0, \quad \zeta^2 \Phi(W) + \Phi(h_-) = 0,$$

Since $f \in G_\perp$, we have $B(f, \mathcal{Y}_\pm) = 0$, which implies that

$$\alpha \zeta \Phi(W) + B(\tilde{h}, h_+) = 0, \quad \alpha \zeta \Phi(W) + B(\tilde{h}, h_-) = 0.$$

Thus we have

$$\Phi(f) = \alpha^2 \Phi(W) + \Phi(\tilde{h}) = -\frac{B(h_+, \tilde{h})B(h_-, \tilde{h})}{\sqrt{\Phi(h_+)}\sqrt{\Phi(h_-)}} + \Phi(\tilde{h}).$$

Step 2. Next we show that

$$h_1 := \Re h_+ \neq 0, \quad h_2 := \Im h_+ \neq 0. \quad (\text{C.7})$$

In other words, h_+ and h_- are independent in the real Hilbert space \dot{H}^1 .

By conclusion (a) in Proposition 2.15, we have

$$L_- \mathcal{Y}_2 = -e_0 \mathcal{Y}_1, \quad L_+ \mathcal{Y}_1 = e_0 \mathcal{Y}_2. \quad (\text{C.8})$$

We show (C.7) by contradiction. First assume that $h_2 = 0$, then from the decomposition of \mathcal{Y}_+ ,

$$\mathcal{Y}_2 \in \text{span}(W),$$

which is the null space of L_- . Thus, $\mathcal{L}_- \mathcal{Y}_2 = 0$, together with (C.8), implies that $\mathcal{Y}_1 = 0, \mathcal{Y}_2 = 0$. But it contradicts the definition of \mathcal{Y}_+ .

Similarly, assume that $h_1 = 0$, and by (C.8), $L_+ \widetilde{W} = 0$ and the decomposition of \mathcal{Y}_+ , we get

$$\begin{aligned}\mathcal{Y}_2 &= \frac{1}{e_0} L_+ \mathcal{Y}_1 = \frac{1}{e_0} L_+ (\xi \widetilde{W} + \zeta W) = \frac{\zeta}{e_0} L_+ W = -2 \frac{\zeta}{e_0} (|\cdot|^{-4} * |W|^2) W, \\ \mathcal{Y}_1 &= -\frac{1}{e_0} L_- \mathcal{Y}_2 = \frac{1}{e_0} \left(\Delta + |\cdot|^{-4} * |W|^2 \right) \mathcal{Y}_2 \\ &= \frac{\zeta}{e_0^2} \left(\Delta + |\cdot|^{-4} * |W|^2 \right) (|\cdot|^{-4} * |W|^2) W.\end{aligned}$$

A direct computation shows that $\mathcal{Y}_1 \notin \text{span}(W, \widetilde{W})$, which contradicts the decomposition of \mathcal{Y}_+ .

Step 3: Conclusion of the proof. Note that Φ is positive definite on $H^\perp \cap \dot{H}_{rad}^1$, we claim that there exists a constant $b \in (0, 1)$ such that

$$\forall X \in H^\perp, \quad \left| \frac{B(h_+, X) B(h_-, X)}{\sqrt{\Phi(h_+)} \sqrt{\Phi(h_-)}} \right| \leq b \Phi(X). \quad (\text{C.9})$$

Indeed it is equivalent to show that, by the orthogonal decomposition on H^\perp related to B

$$b := \max_{\substack{X \in \text{span}\{h_\pm\} \\ X \neq 0}} \left(\frac{B(h_+, X)}{\sqrt{\Phi(h_+)} \sqrt{\Phi(X)}} \right) \left(\frac{B(h_-, X)}{\sqrt{\Phi(h_-)} \sqrt{\Phi(X)}} \right) < 1.$$

Applying twice Cauchy-Schwarz inequality with B , we get $b \leq 1$. Furthermore, if $b = 1$, there exists $X \neq 0$ such that the two Cauchy-Schwarz inequalities become equalities and thus $X \in \text{span}\{h_+\} \cap \text{span}\{h_-\} = 0$, which is a contradiction. Thus $b < 1$.

Now by the coercivity of Φ on $H^\perp \cap \dot{H}_{rad}^1$, (C.6) and (C.9), we have

$$\Phi(f) \geq (1 - b) \Phi(\tilde{h}) \geq c_1 (1 - b) \|\tilde{h}\|_{\dot{H}^1}^2. \quad (\text{C.10})$$

From the decomposition of f , one easily see that $\alpha^2 \Phi(W) + \Phi(\tilde{h}) = \Phi(f) \geq (1 - b) \Phi(\tilde{h})$. Since $\Phi(W) < 0$, we have

$$b \Phi(\tilde{h}) \geq \alpha^2 |\Phi(W)| = \alpha^2 \|W\|_{\dot{H}^1}^2. \implies C \Phi(f) \geq \alpha^2 \|W\|_{\dot{H}^1}^2 + \|\tilde{h}\|_{\dot{H}^1}^2 = \|f\|_{\dot{H}^1}^2.$$

The proof is complete. \square

C.4. Characterization of the real spectrum of \mathcal{L} . Note that \mathcal{L} is a compact perturbation of $\begin{pmatrix} 0 & \Delta \\ -\Delta & 0 \end{pmatrix}$, thus its essential spectrum is $i\mathbb{R}$. Consequently, $0 \in \sigma(\mathcal{L})$ and $\sigma(\mathcal{L}) \cap (\mathbb{R} \setminus \{0\})$ contains only eigenvalues. Furthermore, we have show that $\{\pm e_0\} \subset \sigma(\mathcal{L})$. It remains to show that $\pm e_0$ are the only eigenvalues of \mathcal{L} in $\mathbb{R} \setminus \{0\}$.

We argue by contradiction. Assume that for some $f \in H^2$, we have

$$\mathcal{L}f = e_1 f, \quad e_1 \in \mathbb{R} \setminus \{0, \pm e_0\}.$$

By (2.22), we have

$$(e_1 + e_0) B(f, \mathcal{Y}_+) = (e_1 - e_0) B(f, \mathcal{Y}_-) = 0, \quad e_1 B(f, f) = -e_1 B(f, f).$$

Thus

$$B(f, \mathcal{Y}_+) = B(f, \mathcal{Y}_-) = 0, \quad B(f, f) = 0.$$

Now we write

$$f = \beta iW + \gamma \widetilde{W} + g, \quad g \in G_{\perp}, \quad \beta = \frac{(f, iW)_{\dot{H}^1}}{\|W\|_{\dot{H}^1}^2}, \quad \gamma = \frac{(f, \widetilde{W})_{\dot{H}^1}}{\|\widetilde{W}\|_{\dot{H}^1}^2}.$$

By the coercivity of Φ on G_{\perp} in Proposition 2.15 and Remark 2.16, we have

$$0 = B(f, f) = B(g, g) \gtrsim \|g\|_{\dot{H}^1}^2, \implies g = 0.$$

thus we have

$$e_1 f = \mathcal{L}f = \eta \mathcal{L}(iW) + \gamma \mathcal{L}(W) = 0.$$

Since $e_1 \neq 0$, we have

$$f = 0.$$

This contradicts the definition of the eigenfunctions, and so we conclude the proof. \square

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